THE

CABINET CYCLOPÆDIA.

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A TREATISE
ON
GEOMETRY,
and its Application
To the Arts,
By the Revd. D. Lardner, LL.D.

H. Corbould del. E. Finden

London:
PRINTED FOR LONGMAN, ORME, BROWN, GREEN & LONGMAN'S, PATERNOSTER ROW;
AND JOHN TAYLOR, UPPER GOWER STREET,
1840.

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CONDUCTED BY THE

REV. DIONYSIUS LARDNER, LL.D. F.R.S. L.& E.
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Natural Philosophy.

A

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THE CATENARY.

A

TREATISE ON GEOMETRY,

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APPLICATION IN THE ARTS.

CHAPTER I.

OF STRAIGHT LINES AND PLANE SURFACES.

(1.) The science which in the present advanced state of knowledge under the title of Geometry comprehends so vast and important a field of human inquiry, was at its origin confined in its application to the art of measuring small portions of the earth's surface, and probably had no higher object than to determine the magnitude and fix the limits of property. The annual overflowings of the river Nile obliterated the ordinary boundaries by which the land was subdivided and appropriated, covering the surface with mud. It was therefore necessary to possess some means by which these artificial limits could be from time to time renewed, so that a map of the land being preserved, the property of each person could be re-established. This exigency is said to have directed the attention of the Egyptians to the general properties of geometrical figures; and that as their beautiful relations were gradually developed, the art rose to more noble objects, and was regarded as a subject of higher speculation.

When, however, we consider the multitude of instances of the inevitable application of geometrical principles in the arts of life, even in the first stages of civilisation, it is impossible to conceive that the

discovery or observation of the most simple and obvious properties of geometrical figures could be confined to one country, or could be postponed beyond a very early date in the history of the human race. The natural forms presented by the animal, vegetable, and mineral worlds, the diversified appearance of the surface of the earth, as varied by hill and valley and intersected by seas and rivers, not to mention the equally obvious appearances of the firmament, could not fail to have suggested to the mind the relations of lines and angles, of surfaces flat and curved, and, in short, to have furnished a family of ideas which could not have been long contemplated, without producing some conceptions of general geometrical relations. It may, however, be admitted that such notions may have existed for a period of time, more or less considerable, in a separate and unconnected form, and that the peculiar physical circumstances, incidental to the country of the Nile, united with the early epoch of its civilisation, afford probable grounds for conjecture that these scattered principles, which the constant experience of life must have forced upon every mind, there first received a high degree of generality, and coalesced into a body under the badge of a distinct science. It was, however, after its importation into Greece, that geometry was brought to that state of perfection in which it has been handed down to modern times, having, fortunately, in the works of Euclid and others survived the dark ages.

This science, considered as a part of public instruction, has two distinct objects. First, it may be regarded as an exercise by which the faculty of thinking and reasoning may be strengthened and sharpened. It is peculiarly fitted for this purpose by the accuracy and clearness of which its investigations are susceptible, and the very high certitude which attends its conclusions. Secondly, it is the immediate and only instrument by which almost the whole range of physical investigation can be conducted; without it we could not advance a step beyond the surface of the earth in our knowledge of the

universe; without it we could obtain no knowledge of the figure or dimensions of the earth itself, nor of the mutual mechanical operation, or influence of bodies upon it. In fact there is scarcely a part of natural science in which geometry is not an indispensable instrument of inquiry. According as one or other of these objects have been kept in view, writers on geometry have imparted more or less rigour to their reasonings, and limited their inquiries to topics having more or less immediate application to the arts of life. In the course of instruction followed by the great mass of students in our universities, geometry has been regarded almost exclusively as a system of intellectual gymnastics; while, on the other hand, owing to the very stinted portion of instruction attainable by those who are engaged in the useful arts, the science is with them almost degraded to a mass of rules, without reasons, and dicta, the truth of which is expected to be received on the authority of the writer, and of which the reader is not put in a condition to judge. Such are the extremes of exclusively practical and exclusively theoretical works.

Treatises on this subject, holding an intermediate position, and combining to a considerable extent that rigour of reasoning which has conferred so much beauty and celebrity on the science, with a portion of its useful applications, are less common in this country than in other parts of Europe, where the business of education is conducted with less confined objects. It is our present purpose to endeavour to supply such views of this science as will be found useful to those classes, who while they do not pursue geometry as a mere intellectual exercise, are capable, nevertheless, of appreciating its clearness and certainty, and are unwilling to receive a proposition as true without a proof of it, where a proof may be obtained; and who, on the other hand, also delight to contemplate some of the most important useful purposes to which the abstract principles of the science have been applied.

There is no part of geometry which has given rise

to so much and so unprofitable discussion as the formal explanation of those terms which express the primary notions involved in geometrical investigations. According to the rigorous method of treating of the science, it has been thought indispensable to lay down in the first instance certain formal definitions of the objects or notions which constitute the subjects of investigation, and from those and certain propositions called axioms to deduce all the conclusions of the science.

The meaning of a term may be made known in either of three ways:— First, by another term synonymous with it, the import of which may happen to be better understood; Secondly, by shewing the object or thing signified by the term to be explained; Thirdly, by a sentence composed of several terms not synonymous with each other, but signifying collectively the meaning of the term to be explained.

It is the last alone which can be properly called a definition. A synonymous term may not be better understood than the term to be explained, and will itself stand equally in need of definition. To show the object will be effectual, when an object can be found which is a strict representative of the term in question. This, however, is not always the case. The explanation of a term by several other terms not synonymous with each other, is applicable only to terms expressing compounded notions, and cannot have any application to terms of simple and uncompounded meaning, because the several terms of which such a definition is composed, signifying many different conceptions of the mind, cannot represent a term which signifies one uncompounded conception.*

It is obvious that definition must stop somewhere. Since one term can only be defined by other terms, these others themselves must be defined; and it is clear that we must ultimately come to a term, the meaning of which must be obtained by some means independent of mere language. Now it so happens, that all these

* See Locke on the Human Understanding, Book III.

difficulties attending the process of definition, are especially involved in the explanation of the terms which form the basis of geometrical reasonings. Many of them are names of conceptions so abstract, that no actual object existing presents a precise representation of them; and they are conceptions so uncompounded that they do not admit of being explained by a combination of other terms.

A point, a line, a surface, a solid, a straight line, a curved line, are among the terms the meaning of which is necessary to be understood in the very commencement of geometrical inquiry; yet there is scarcely one of them which admits of being explained by other terms. The object of a definition is to make the meaning of a word understood which was before unknown; and it will scarcely be denied, that a definition which fails to accomplish this is useless. It is evident that the terms of a definition should be better understood than the terms which they define; and that their combination, when rightly understood, should precisely and clearly signify that which the term they define is designed to express. Let these tests be applied to the following definitions: —

A point	- -	a monad having position. — *Pythagoras.*
Ditto	- -	that which has no parts. — *Euclid.*
A line	- -	length without breadth. — *Euclid.*
A surface	- -	length and breadth only. — *Euclid.*
A straight line	-	that which lies evenly between its ends - - — *Euclid.*

In fact, these and many other terms of current use in the elements of mathematical science, neither admit nor require strictly logical definitions. If the accomplished geometer retraces the steps by which he has himself acquired clear and distinct notions of them, he will find that such conceptions have been the result, first, of observation of material objects; and, secondly, of those processes of mental reflection upon them by which the first rude notions derived from sensible objects are modified and corrected.

(2.) The common popular notion of a point is derived from the sharpened extremity of any long and narrow body, such as the end of a fine pin or needle. This, supposing it to be the smallest magnitude perceptible by the senses, is called a *physical point:* if this point were indivisible, even in imagination, it would be a mathematical point: but this is not the case. No material substance can assume a magnitude so small that a smaller may not be conceived. The point of the finest needle, the extremities of the thinnest hair, the ends of the most delicate fibres of cotton, silk, or spider's web, are extremely minute magnitudes, and in the loose application of language in ordinary topics of investigation, such magnitudes may not improperly be called points. But it is easily demonstrated, that even the smallest of these has definite magnitude, so that it is divisible; and, therefore, a still smaller magnitude may be contemplated. Now, a mathematical point utterly precludes the possibility of subdivision; length, breadth and thickness, are attributes altogether inapplicable to it: it possesses no quality of magnitude, and nothing can be stated respecting it *per se,* except that it has a certain assignable position in space. These considerations will throw some light on the Pythagorean definition by which a mathematical point is declared to be "a monad* which has position."

The rigour of the ancient geometry excluded the idea of motion; and the elements of the science were thus deprived of one of the most useful instruments of illustration and reasoning. In a treatise such as the present, it is not necessary to restrict our method by rules so severe, and we shall freely use such illustrations and such modes of reasoning, as may appear best suited to convey to the minds of ordinary readers clear conceptions of the objects with which the science is convers-

* From the Greek word μονὰς, which signifies unity, singleness, or indivisibility. This definition, therefore, only adds the positive quality of having position to the negative quality of the absence of parts expressed in Euclid's definition.

ant, and as will best render manifest the truth of its most important conclusions.

(3.) If a mathematical point be conceived to move through space, and to mark its course by leaving behind it a trace or track, that trace or track will be a mathematical line.

In like manner, if a physical point be conceived to move, its trace or track will represent a physical line.

As a physical point is only an extremely minute magnitude having some dimensions, however small, its trace or track will evidently have corresponding dimensions. A physical line, therefore, has breadth and thickness corresponding to the magnitude of the physical point by the motion of which it is conceived to be produced.

An extremely fine thread or fibre may be considered as affording an example of a physical line.

But, as a mathematical point has, strictly speaking, no dimensions, even in idea, its trace or track can have no dimension but that of length. To suppose that its track has breadth or depth, would involve the supposition that the point itself has dimensions corresponding to this breadth and depth, which is contrary to what has been stated, respecting such a point.

It is clear, therefore, that whatever qualities may belong to a mathematical line, it has neither breadth, depth nor thickness, nor any other dimensions except length.

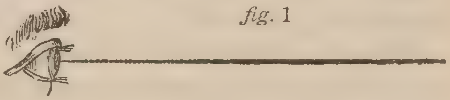
If a mathematical point move continually in the same direction, its track is called a straight line or a right line; if, on the other hand, it continually change its direction as it moves, its track is called a curved line or a curve.

Much controversy has been maintained among geometers respecting the definition of a straight line. To the explanation just given, that it is produced by the motion of a point proceeding in the same direction, it is objected, first, that the idea of motion is not necessarily connected with that of a line; and, secondly, that the words

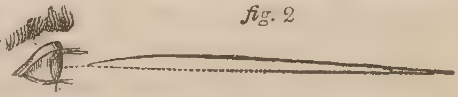
"same direction" have no other meaning than the words straight line, and that, therefore, they stand as much in need of definition as the terms which they are used to define.

To Euclid's definition, that a straight line lies evenly between its extremities, it is objected that the term "evenness" can have no other import than straightness, and that, therefore, the definition is merely the substitution of one term for another, the term substituted standing as much in need of definition as the term defined.

Plato defined a straight line by a certain optical property which characterises it, and which belongs to no other line. If the eye be placed in such a position beyond its extremities that one end of the line shall conceal from the sight the other end, then every part of the line between the extremities will also be hidden. *Fig.* 1.



It is obvious that a curved line would not possess this property; and that, on the contrary, if one of the extremities of such a line were placed between the eye and the other extremity, more or less of the intermediate part of the line would be in view. *Fig.* 2.



The definition of a straight line given by Archimedes, and subsequently by many later geometers, is, that it is the shortest way between its two extremities.

If a light and flexible string be extended by drawing its extremities from one another, it will assume, between the points of tension, a certain position. Speaking without the rigorous exactitude of geometry, it might be called a straight line; but since it is evident that the string has weight, that weight must be admitted to produce some flexure, the convexity of which will be presented downwards; and to whatever extent this flexure exists there will be a corresponding deviation

from the quality which essentially characterises a straight line. If the thread, however, be imagined to be altogether deprived of weight, which it would be if the earth were removed from it, then it would take a position between the two points of tension free from all flexure, and would be accurately a straight line.

It is evident that this view involves the quality included in the definition of Archimedes, that it is the shortest distance between two points.

(4.) If a curved line, such as A B, *fig.* 3., be conceived to turn round its extremities, as fixed points or pivots, and, as it turns, to leave behind it a trace or track, that trace or track would include a certain portion of space. This space would be round in its form, taken in a transverse direction, and would be such as is represented in *fig.* 4.

fig. 3

A  B

This is a circumstance which is common to every curved line which can pass between two points: every such line by its revolution round its points, must enclose more or less space.

fig. 4

A  B

A straight line is the only line which can never be attended with this effect. If it be conceived to turn round its ends considered as fixed, it will not, as it revolves, include any space.

It will much contribute to the clearness of the notions of a student, if a piece of wire bent into the form of a curve be made to revolve round its ends, so that it will enclose space; but if the same wire be straightened, and submitted to the same operation, the same effect will not be produced.

This property of a straight line is the subject of one of the axioms prefixed to the first book of Euclid's Elements, and is expressed thus: "Two straight lines cannot enclose a space."

The same character of straight lines may also be expressed, by stating that two straight lines cannot meet

in more than one point: for if they met in two points, and did not coincide in all the intermediate points, they must evidently enclose a space. On the other hand, if they *did* meet in *all* the intermediate points they would be one and the same straight line, and not two different lines.

It is evident that when a flexible string is stretched tight between two points it takes a definite position between them; and that two different strings stretched between the same points, would not take different positions, and therefore could not enclose space.

This property belongs to straight lines exclusively, and is not shared by curves. If two flexible threads hang loosely between two points, they may be very far asunder, and may, consequently, enclose space.

When the moon is new its edges form two curved lines between its horns, inclosing the enlightened part of the moon.

(5.) Perhaps the clearest notion of a surface will result from the consideration of the external limits of a solid body. A surface is defined in geometry to be that which has the positive attributes of length and breadth, and the negative quality implied by the absence of depth or thickness. If the external limits of a solid be taken as the meaning of the term surface, it is evident that it excludes the notion of depth, since any portion of depth, however small, which might be assigned to it, would necessarily penetrate within the external limits of the solid.

But the geometer requires that a clear conception of a solid should be formed, independently of the presence of a body: thus, a surface may be conceived to exist in space, from which all matter may be excluded. The earth moves round the sun in a certain path, and within that path is included a certain surface between it and the sun. Now, this surface must be contemplated and reasoned upon, even though it should be denied that any material substance exists on either side of it.

The notion of a mathematical surface may be formed by imagining a mathematical line to move in any manner in space, leaving behind it, as it moves, a trace or track. This trace or track will be a mathematical surface; and, as the line by whose motion it was produced, has no thickness, it is clear that the surface can have no depth.

It is also evident that the limits or edges of the surface will be mathematical lines, for its extreme boundaries will be the initial and final positions of the mathematical line, by whose motion it is generated, and the other edges will be the lines traced by the extremities of that line as it moves.

The definition of a plane surface has been attended with difficulties similar to those which we have described, in reference to the definition of a straight line.

Euclid's definition of a plane surface is, "that which lies evenly between its extremities." This is subject to the same objection as that which is advanced against the corresponding definition of a straight line.

A plane surface has also been defined by a method analogous to Archimedes' definition of a straight line, to be the smallest surface that can be included between given extremities.

Plato defined a plane surface by a process analogous to that which he adopted as the basis of his definition of a straight line. He explained it by stating it to be a surface, one of whose extremities will hide every part of it, the eye being placed in its continuation. This is an optical property which characterises a plane surface, and belongs to no other.

If any two points be taken in a plane surface, and a straight line be drawn joining them, every point of that straight line will be in the plane surface; and if the same straight line be continued beyond the points which it unites until it meets the extremities of the surface, every part of its continuation will likewise be in the plane surface. This will be the case with every

straight line whatever, which can be drawn joining any two points in a plane surface. This is a property which belongs exclusively to plane surfaces, and which does not appertain to curved ones.

There are certain curved surfaces in which it is possible so to select two points, that every part of a straight line joining them shall be in the curved surface. The difference, however, between these and a plane surface is that in the curved surface, the points which possess this property must be selected in a particular manner upon the surface, whereas, in a plane surface, it belongs indifferently to every two points which it is possible to assume.

This property is used in the arts as the test by which a surface is determined to be plane, and the analogous property of a straight line is similarly adopted as a test of straightness. If two straight lines are made to coincide in any two points we have shewn that they will coincide in every point, as well as in those between the two points as in those beyond them. Hence the perfect straightness of a line, is determined in the arts by taking a ruler having a straight edge, and placing any two points of that edge upon two points of a line whose straightness it is designed to examine. If every other point of the line is found to coincide with the edge of the ruler the line will be straight, but otherwise not.

If it be desired to determine whether any proposed surface be plane, let any two points of the edge of the same ruler be placed upon two points of the proposed surface, and observe whether every part of the edge of the ruler in that case touches the surface. If it do not, the surface cannot be plane. If it do, then change the position of the ruler so as to give it another direction upon the surface, and make the same observation. If it be found that in every position which can be given to the ruler its edge will coincide with the surface in every point, then it may be concluded that the surface in question is a plane surface.

That a straight edge may in certain positions coincide with a surface which is not plane, will be readily understood. If such an edge be applied to the surface of the shaft of a round pillar, it will coincide with the surface, provided it be applied to it in the direction of the length of the pillar ; but if the edge be applied to the same surface in any transverse direction, it can only touch the surface in one point.

If the same edge be applied to the inner or concave surface of the arch of a bridge, it will coincide with it in every point, provided it be applied in the direction of the length of the arch ; but such a coincidence cannot take place, if it be applied in any transverse direction.

The Platonic definitions of a straight line and plane, founded upon their optical properties, are tests of straightness and evenness, which are also commonly used in the arts. To apply the tests just adverted to, it is necessary that we should possess an edge which is itself perfectly straight, and some independent test of the straightness of such an edge is therefore necessarily supposed. Such straightness is usually determined by holding the edge before the eye, and looking along it in such a manner that the nearer extremity shall be precisely between the eye and the remote extremity. If any intermediate part of the edge be above or below, or to the right or to the left of the direction of sight, it will be immediately perceived.

The test of straightness adopted as the definition of a straight line by Archimedes, is also used extensively in the arts. In ornamental horticulture straight lines forming the edges of paths and roads, or rows of plants and shrubs, are determined by stretching a flexible cord between their extremities. In architecture, the straightness of the upright faces and corners of buildings is determined by the direction of a flexible cord stretched by a weight. In carpentry and other arts, straight lines are described upon plane surfaces by stretching between two points upon the line sought to be described, a flexible cord previously rubbed over with chalk ; when a

sufficient tension is given to it, it is raised with the hand, and is allowed to recoil upon the surface by its elasticity, leaving a chalked mark indicating the direction of the required line.

A plane surface is often produced in the arts by the motion of a straight line. The action of a carpenter's plane is founded upon this principle. That instrument consists of a cutter with a perfectly straight edge, which projects slightly from a slit in the frame in which it is fixed. The edge of this cutter being moved over the surface which is required to be rendered plane, shaves off all its projecting asperities, and if there be any parts more hollow than the rest, it continually reduces the general surface, until the edge of the cutter in every part of its motion is in contact with the surface.

The more refined is any art which involves the principles of design and the greater the accuracy required in its productions, the more nearly do the practical lines and surfaces approximate to the perfect precision of the abstract geometrical conceptions; and nothing can more conduce to the advancement of practical skill in the higher departments of the useful arts, than the early cultivation of pure geometrical principles by the artisan. The edges and surfaces produced in house-carpentry are rude attempts to imitate the perfection of the mathematical conceptions of straight lines and plane surfaces. Between one artisan and another however, even of this class, there are comparative degrees of skill, and the better workman always approaches more near to geometrical precision: his edges are more truly straight, and his surfaces more truly plane. Nor is the end attained by such skill merely the production of external beauty. Stability of structure depends as much as external grace on such precision; but the higher we ascend in the arts, the more nearly do we approximate to geometrical perfection. The designs of the engineer and the machinist are often attended with an exactitude truly admirable, and subject to deviations from geometrical accuracy scarcely more than micro-

scopic. In their designs lines are expressed of a breadth only sufficient to be visible, and of beautiful uniformity, approaching as near to the spirit of a geometrical line as it is possible for a production of art to imitate a creature of the mind. The advantages, therefore, accruing to artisans from a due cultivation of the principles of pure geometry, are not confined merely to invigorating their discursive powers, nor to storing their memories with principles of art useful in almost every department of their daily occupations; but in addition to these important purposes, such a study inspires a taste for that precision of construction, and a love for that accuracy of form, in the absence of which no artisan or engineer, whatever be his grade, can hope to arrive at great professional excellence.

CHAP. II.

OF ANGULAR MAGNITUDE.

(6.) When two or more straight lines are drawn from the same point, they will have different directions, and any two of them may differ more or less in this respect; lines thus emanating or radiating from a common point are said to *diverge* from that point, and the quantity of the *divergence* of any two of them is expressed by the word *angle.*

Angles may not improperly be considered as a species of magnitude, since they are as capable of being expressed by number as the other modes of magnitude.

To illustrate the nature of angular magnitude, let C be supposed to be the extremity of a straight line extending indefinitely in the direction C A. Through the same point C, let another indefinite straight line C A_0, be conceived to be drawn, and suppose this latter line to revolve round its extremity C, being supposed at the beginning of its motion to coincide with the fixed line C A. As the line C A_0 revolves, it will take successively the positions marked by C A_1, C A_2, C A_3, C A_4, &c. and will in this manner make a complete revolution round the point C. When it has made half its complete revolution, it will take a position C A_5, precisely opposite to its first position C A_0, so that the two lines C A_0 and C A_5, shall form one continued straight line.

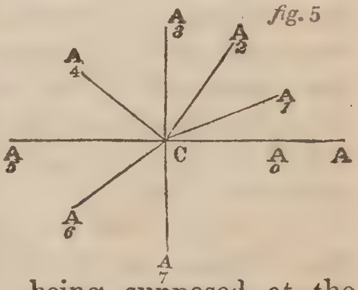


fig. 5

When it has performed one fourth of its complete

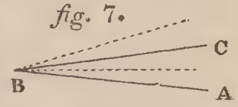
revolution, it will have a position $C A_3$, at equal angular distances between $C A_0$ and $C A_5$.

(7.) The angle which the line thus revolving forms in any one of its positions, with the fixed line C A, is not affected by either the length of C A or the length of the revolving line. These lines are called the *sides* of the angle, and the point C where the sides unite, is called its *vertex*.

(8.) The equality or inequality of two angles is determined by a process called *super-position*, which is of extensive use in elementary geometrical reasoning. Thus, if it be wished to determine whether the angle A B C (*fig.* 7.) is equal or unequal to the angle A′ B′ C′ (*fig.* 6.), it will only be necessary to place the point B′ upon the point B, and side B′ A′ upon the side B A, and to let the side B′ C′ fall upon the plane of the angle A B C. If under these circumstances the line B′ C′ shall be found to lie upon the line B C, then the angle A′ B′ C′ will be equal to the angle A B C; but if the line B′ C′ should fall *below* the line B C, then the angle A′ B′ C′ will be less than the angle A B C. If, on the other hand, the line B′ C′ should fall *above* the line B C, then the angle A′ B′ C′ will be greater than the angle A B C.

fig. 6.

fig. 7.



(9.) It is usual in geometry to express an angle by three letters, one of which is at its vertex, and the other two at any points upon its sides. As the magnitude of an angle does not depend upon the length of its sides, it is immaterial what position upon the sides the latter letters may have. In expressing the angle, however, the letter at its vertex is always placed in the middle. When the same vertex belongs only to one angle, the angle may be expressed by the vertical letter alone without the lateral ones. Thus in *fig.* 6. the angle which we have called A′ B′ C′ might also be called the angle B.′

But where two or more angles have a common vertex, it is necessary to express each of them by three letters;

thus, in *fig. 5.*, there are several angles, of which the point C is the common vertex; and, in this case, each angle must be expressed by the two letters which mark its sides with the letter C which marks its vertex between them.

(10.) In *fig. 5.*, if we suppose the paper to be folded over, so that the line C A shall be precisely doubled down upon the opposite line $C\,A_5$, which is its continuation, the fold of the paper will evidently take the direction which the revolving line would have after it has completed one fourth and three fourths of its complete revolution; for the angle between this fold and $C\,A_0$ will be superposed upon the angle between the same fold and $C\,A_5$, and will therefore be equal to it; and in the same manner the angle included between C A and the lower part of the fold, will be superposed upon the angle between $C\,A_5$ and the lower part of the fold, and will therefore be equal to it. The fold of the paper will therefore divide each half revolution into two equal parts, and will therefore divide one entire revolution into four equal parts.

(11.) The angle $A_0\,C\,A_3$, which forms the fourth part of a complete revolution, is called a *right angle*, and it is manifest that the four right angles formed round the point C by the lines $C\,A_0$, $C\,A_3$, $C\,A_5$, and $C\,A_7$, are equal to each other.

It is also evident that the line $C\,A_7$ is only the continuation downwards of the line $C\,A_3$, since both coincide with the fold of the paper.

(12.) All the angles which can be formed by diverging lines, however numerous, round a common centre C, will always make up, when added together, a sum of four right angles; this must be manifest, since the angular space which they fill, is the same as that filled by the four right angles which surround the point C.

(13.) Angular magnitude is expressed numerically by dividing the space surrounding the point C into a number of equal angles by diverging lines, and giving these angles some common denomination. The ancients, and, for the most part, also the moderns, suppose 360 lines to diverge from the common centre C, forming

with each other equal angles. Each of these angles is called *a degree.* Thus 360 degrees make up four right angles, and therefore 90 degrees make one right angle. It is usual to express degrees by placing an ° over the number. Thus, 360° signifies 360 degrees, and 90° signifies 90 degrees. A different division of angular magnitude has been introduced and partially adopted in France. French mathematicians conceive the angular space surrounding the centre C to be divided into 400 equal angles: each of these angles is called a degree, and a right angle, therefore, consists of 100 degrees.

The decimal or centesimal division of angular magnitude is attended with some convenience in numerical calculations which the sexagesimal does not possess; but, on the other hand, the sexagesimal division is attended with other advantages which the decimal wants. There are several angles of particular magnitudes, to which frequent reference is necessary in geometrical and physical inquiries, and there is great convenience in being enabled to express such angles by whole numbers. The sexagesimal division allows this by the great variety of whole numbers which are exact divisors of 360. The integral divisors of this number are the following: — 180. 120. 90. 72. 60. 45. 40. 36. 30. 24. 20. 18. 15. 12. 10. 9. 8. 6. 5. 4. 3. 2.

On the other hand, the integral numbers which exactly divide 400 are only the following: — 200. 100. 80. 50. 40. 25. 20. 16. 10. 8. 5. 4. 2.

(14.) As we shall, in accordance with the universal practice of English writers, use the sexagesimal notation for angles, it will be convenient that the student should be familiar with the numerical denominations for certain angles to which we shall have frequent occasion to refer, among which the following may be mentioned: —

A right angle - - -	90°
Two right angles, or two lines in continuity	180°
Three right angles - -	270°
Half a right angle - - -	45°
Two thirds of a right angle - -	60°
One third of a right angle - -	30°

(15.) When an angle is less than 180°, the quantity by which it falls short of that amount is called its *supplement.* Thus the supplement of 45° is 135°, the supplement of 60° is 120°, the supplement of 30° is 150°, and the supplement of 90° is 90°.

(16.) When an angle is less than a right angle, the quantity by which it falls short of a right angle is called its *complement.* Thus the complement of 45° is 45°. The complement of 30° is 60°.

(17.) An angle which is less than a right angle is said to be *acute,* and an angle which is greater than a right angle is said to be *obtuse.* In other words, all angles between 0° and 90° are acute, and all angles between 90° and 180° are obtuse.

The supplement of an acute angle is obtuse, and the supplement of an obtuse angle is acute.

The nature and properties of angular magnitude, and the terms and numbers by which it is expressed, are of the most extensive use in the sciences and in the arts. In some cases, as in astronomy and geography, it is by angular position almost exclusively that the actual distances and local arrangement of the objects of enquiry are determined. The real distances of the numerous luminaries which so richly furnish the firmament can be discovered by no other means than an elaborate and accurate determination of their *apparent* positions.

The *apparent position* of an object is a term used in science to express the position of the object so far as it can be determined by the sight. It is angular position only of which the sight can form an estimate. Two distant objects may be seen in juxtaposition : their angular separation may be perceived by the sight ; and if the sight be assisted by proper metrical instruments, their exact angular separation may be numerically determined. But this is obviously a result altogether independent of their actual position in space. Their angular or apparent distance apart may be exceedingly small, or may even be nothing, while their actual distance may extend to any degree of magnitude. The sun and moon are frequently seen in the heavens separated from each

other by but a small apparent distance: that apparent distance is measured by the angle contained by two straight lines drawn from the eye of the observer to the centres of those objects, without any regard to the length of those two lines. It sometimes happens that these two lines are confounded together, and include no angle, the line from the eye of the observer, which passes through the centre of the one luminary, also passing through the centre of the other, and the two luminaries having the same apparent position. But it will be evident that in all these cases the real position of those bodies in the universe is exceedingly different; the distance of the centre of the moon from the eye of the observer being about 240,000 miles, while the distance of the centre of the sun is not less than 96,000,000 of miles.

It is by the observation of the visual angles under which distant objects are seen, that surveys of the earth's surface are executed. It is likewise by similar means that the navigator coasting along a known country, determines from place to place the position of his vessel on the trackless surface of the waters, by observing the *bearings* of various landmarks. It is by such means that the dangerous position of sunken rocks is made known to him. It is by such means that the boundaries of shoals and sand banks are as clearly delineated upon the fluid surface of the deep, as the geographical boundaries of the divisions of land are marked upon the soil by permanent and visible natural or artificial limits.

In the useful arts, cutting tools of every description have their edges formed into angles of various magnitudes, according to the materials on which they are intended to act. In general, the softer the material to be divided, and the more accurately the separation is to be effected, the smaller will be the angle of the tool. Chisels for cutting wood, are formed at angles of about 30°; those for iron, at from 50° to 60°; and those for brass, at from 80° to 90°. In general, chisels intended to act by pressure, may be constructed with angles more acute than those which act by percussion; the edge in

the one case requiring more strength to resist fracture than in the other. Knives intended for the division of soft substances in domestic economy, are constructed with extremely acute edges, since they are intended to act by pressure, and are not usually submitted to any violent action.

An extreme example of an acute edge is that of the razor. This instrument, according as it is used, may act either as a chisel or a saw: if it be made to remove the beard by a motion perpendicular to the direction of its own edge, its action will be that of a chisel; but if its edge be oblique to the direction of its motion, or, what is the same thing, if while it is advanced perpendicular to its edge, it is likewise drawn from heel to point, it then acts as a saw.

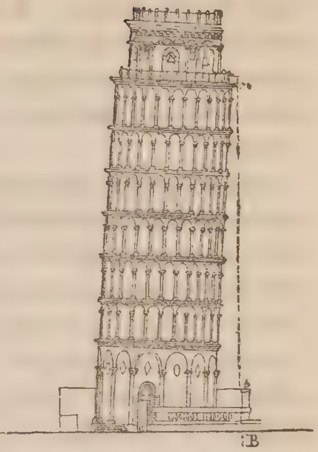
The same observations may be applied to all the sharper classes of cutting instruments.

(18.) The angle which is by far the most extensively used in the arts, is the right angle, chiefly because it is the angle of mechanical equilibrium, between the direction of any impact or pressure, and the surface resisting it. A force cannot be entirely counteracted by any surface, unless that surface be exactly perpendicular to the direction of the force.

On the other hand, if it be desired to produce an effect by a force upon a surface, either by compressing, breaking, or penetrating it, the force cannot be perfectly efficient for such a purpose, if its direction be not perpendicular to the surface.

It is this principle which fixes the relation between the direction of gravity, and all surfaces destined to sustain weights; it is this principle, which determines the erect position of the natural structures of animals and plants; it is this which confers majesty and beauty upon the forest and the mountain; and it is by following out the architecture of nature, that artificial structures raised by the hand of man acquire stability and beauty. Buildings are erect, because the direction of their weight must be perpendicular to its support; and the violation of this law in particular cases, as in the

fig. 8.



leaning tower at Pisa (*fig.* 8.), inflicts instant pain on the beholder. A steeple or tower, which, by the yielding of its foundation, or any other cause, is out of the perpendicular, cannot be beheld without some sense of danger, and consequently some feeling of pain.

(19.) The instrument called the square used in the arts, is a model or pattern of a right angle, by which right angles may be delineated in drawings, or formed in structures This instrument consists of two flat rulers as represented in (*fig.* 9.), which ought to be placed so that their edges, both internal and external, may be precisely at right angles with each other. When much accuracy is desired, great care should be taken in the selection of such an instrument. Of the numerous squares offered for sale, and used by artisans, comparatively few either have, or if they have, retain a very high degree of precision. It is easy to test the accuracy of an instrument of this kind. Let an angle A B C. (*fig.* 10.) be drawn with it, and continue the side C B to D so as to form the angle A B D the supplement of A B C. If A B C be exactly a right angle, then its supplement A B D should also be exactly a right angle (15.).*

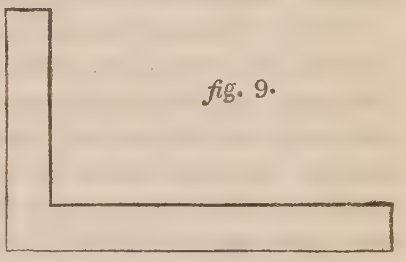


fig. 9.

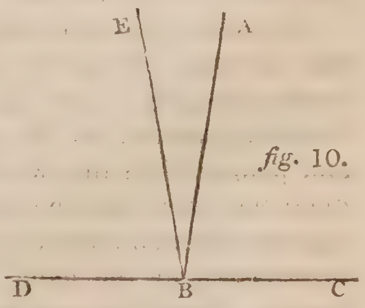


fig. 10.

* Where any statement depends on, or is inferred from, a former article,

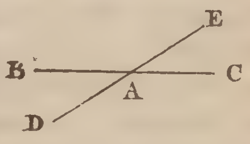
If therefore, upon applying the square to the angle A B D, it should be found exactly to correspond with it, the square will be correct; but if the angle of the square be less than the angle A B D, it will be less than a right angle, and if it be greater than the angle A B D, it will be greater than a right angle.

By this process we can not only determine whether the square be exactly formed, but if it be not so formed, we can determine the amount of its error, or the magnitude of the angle by which it exceeds or falls short of 90°.

Let us suppose that, upon applying one edge of the square to the line B D, the other edge, instead of coinciding with the line B A, is found to take the direction B E. It is evident that the three angles C B A, D B E, and E B A, make up together 180°; but the angles A B C and D B E are each equal to the angle of the square. If, therefore, the angle E B A be taken from 180°, the remainder will be twice the angle of the square; and if half of E B A be taken from 90°, the remainder will be the angle of the square. The angle, therefore, by which the square falls short of 90°, will be half the angle E B A.

In the same manner it may be shown, that if the line B E fell within the angle A B C, the angle of the square would be too great by half the angle included between B A and B E.

fig. 11.



(20.) When two straight lines cross each other, as in *fig.* 11. the angle B A D is said to be *vertically opposite* to the angle E A C, and, in like manner, the angle B A E is vertically opposite to D A C.

When two straight lines thus intersect each other, the angles which are vertically opposite are equal, for if the angle B A E be added to the angle E A C, the sum will be 180° (14.); and if the same angle be added to the

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the reference will be made by merely annexing the number of the article from which the inference is made, as in this case it follows from (15.)

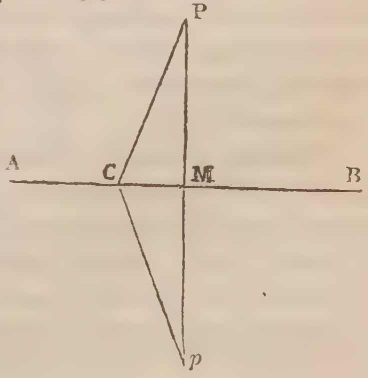
angle B A D, the sum will likewise be 180°. Hence the angle E A C must be equal to the angle B A D.

In the same manner, if the angle B A D be added either to D A C or B A E, it will give a sum of 180°, and, consequently, the angles D A C and B A E are equal.

(21.) If from any proposed point P (*fig.* 12.), several straight lines be drawn to a given straight line A B, and if one, P M, of these straight lines be perpendicular to A B, it will be shorter than any of the others. Let P C be any one of the others, and suppose P M continued below A B until M *p* shall be equal to M P, then let a straight line be drawn from C to *p*; now if we suppose the paper folded over so that the line M *p* shall lie upon the line M P, the fold of the paper will correspond with the line A B, because the angle P M B is equal to the angle *p* M B; and since the line M P is equal to the line M *p*, it is evident that the line C *p* will precisely cover the line C P, and therefore must be equal to it. Now since a straight line is the shortest distance between two points, P M *p* will be less than P C *p*, and consequently P M which is half the former will be less than P C which is half the latter; and in like manner the line P M may be proved to be less than any other line which can be drawn from P to the line A B.

(22.) That only one perpendicular can be drawn from a given point P, to a straight line A B, is a proposition so nearly self evident that it admits of no other kind of proof but that which consists in showing that any thing contrary to it must be absurd. If it be admitted, for a moment, to be possible that a second perpendicular could be drawn, let the line P C, *fig.* 12., represent that perpendicular, and, as before,

fig. 12.

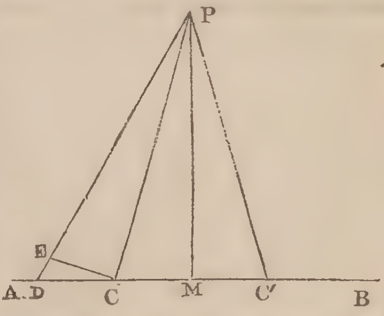


draw C p; by the same process of folding back the figure, it may be shown that the angle p C M is equal to P C M, because the one exactly covers the other. But since P C is here supposed to be perpendicular to A B, the angle P C M is a right angle, therefore p C M must also be a right angle; and this being the case, P C p must be one continued straight line: but P M p is also one continued straight line. Thus there would be two different straight lines joining the same points, P p, which is contrary to what has been already explained (4). Hence, the supposition of the possibility of drawing from a point to a straight line more than one perpendicular, involves an absurdity.

(23.) From this reasoning it immediately follows, that if from any two points in a straight line two lines be drawn both perpendicular to that straight line, these lines can never meet, for, if they did, then they would, in fact, be two perpendiculars drawn from the point where they would meet to the same line, which is contrary to what has been just demonstrated.

(24.) If several lines be drawn from the same point, P (*fig.* 13.), to the same straight line, A B, one of which is perpendicular to it, those lines will be equal which meet A B at points equally distant on different sides of the perpendicular, and the more distant from the perpendicular the points are at which such lines meet the line A B, the longer will such lines be.

fig. 13.



Let P M, as before, be the perpendicular, and take M C equal to M C′, the lines P C and P C′ will be equal; for if the paper be folded over along the line P M, the line M C′ will fall upon the line M C, because the angle P M C′ will be equal to the angle P M C, and the point C′ will fall upon the point C, because the line M C′ is equal to the

line M C ; since then the point C′ falls upon the point C, the line P C′ must coincide with the line P C, and therefore they must be equal.

Let D be a point on A B more distant from M than C is. We are then to prove that the line P D must be greater than P C. Suppose a line C E, drawn from C at right angles to C P ; since P C is perpendicular to C E it will be less than P E (21.), but P E is less than P D, therefore P C is less than P D, and in the same manner it may be proved that the more distant any line is from the perpendicular P M, the greater it is.

(25.) The same process of investigation will easily show, that the lines drawn from P to points equally distant from the perpendicular are inclined at equal angles to the line A B, and that they are also inclined at equal angles to the perpendicular P M. It will also follow, that the more remote the lines are from the perpendicular, the less will be the angles at which they are inclined to A B, and the greater will be the angles at which they are inclined to the perpendicular.

(26.) It is obvious that the lines more distant from the perpendicular will make greater angles with it ; but it is not, at first view, so apparent that they will make less angles with the line A B. In *fig.* 14. let the lines

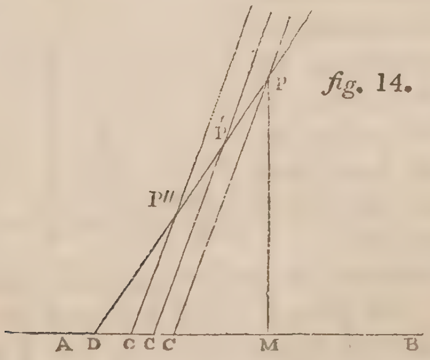


fig. 14.

C P and D P be continued beyond the point P, and let the angle M C P be imagined to be moved towards the point D, C M still remaining upon the line A B. It is evident that as the angle is thus moved, the point

where its side crosses the side of the angle M D P, will move from its present position towards D, taking successively the positions P′, P″, &c.; the length of that portion D P′, D P″, &c. of the side D P, which is contained within the angle M D P, will gradually diminish, and when the angle C is moved to D its side will lie altogether above the side of the angle at D, and therefore the angle C must necessarily be greater than the angle D.

(27.) For the construction of a *square*, or the model of a right angle, it is necessary that we should be able to delineate an exact right angle by which such a square may be made. The preceding principles indicate a method of accomplishing this.

Having drawn any straight line, such as A B (*fig.* 13.), take any point, M, upon it, and on each side of M take equal distances, M C, M C′. Find a point, P, which shall be equally distant from C and C′, and draw a straight line from this point P to M. That line P M will be perpendicular to A B.

The greater the distances M C and M C′ are taken, the more accurately will the position of the perpendicular be defined.

CHAP. III.

OF PARALLEL LINES.

(28.) It was shown in the last Chapter (23.) that if two straight lines be drawn from any two points upon a given straight line, both perpendicular to it, they can never meet, to whatever distance they may be drawn. Two such lines are said to be *parallel.*

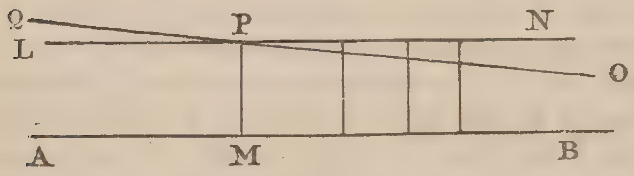
The doctrine and properties of parallel lines have always held a conspicuous place in geometry, and have been the more remarkable, in that no geometrical skill has ever succeeded in reducing their investigation to the same simple and fundamental principles, which have always been considered as conferring the last degree of precision and clearness on the investigations of elementary geometry. Even the most remote and difficult propositions in other parts of the science, are deduced by rigorous demonstration from certain general axioms admitted to be so clear in their nature, that their demonstration, or their deduction from other more simple and evident truths, is equally unnecessary and impossible. But it has been the reproach of geometry, that the theory of parallel lines has never been established, without either introducing among the axioms some proposition whose truth is less evident than that of many other propositions of geometry already admitted to be capable of, and to require proof; or by introducing methods of investigation, deficient in the rigour and foreign to the spirit which characterises every other part of elemetary geometry.

Probably the origin of this difficulty may be traced to the very nature of parallels, and the hopelessness of

surmounting it may be thereby made manifest. It is found that in every case where the notion of infinity finds its way into mathematical inquiries, artifices of reasoning of a peculiar kind must be resorted to. Those who are conversant with the higher analysis, are familiar with this fact. Now parallels cannot be defined or understood so as to exclude the notion of infinity. Euclid defines them to be lines which, being continually produced in both directions, can never meet; the meaning of which is, that though they be infinitely prolonged, they cannot cross each other.

(29.) The fact already established, that straight lines which are perpendicular to the same straight line can never meet (23.), leads to the solution of the problem to draw through a given point a straight line which shall be parallel to a given straight line. Let P (*fig.* 15.) be the point through which it is required to draw a straight line parallel to A B.

fig. 15.



From P suppose P M drawn at right angles to A B, and then let a straight line L N be drawn through P perpendicular to P M. Since L N and A B are both perpendicular to P M, they are parallel to one another by what has been already proved (23.), therefore L N is a parallel to A B, through the point P.

(30.) The several properties of parallel lines which now remain to be established, cannot be deduced from what has been proved without assuming some one of them without demonstration. That which we shall assume is, that through the same point only one parallel to the same straight line can be drawn. This appears to be on the whole the principle connected with parallels, which the mind admits most readily without proof,

and its admission will enable us to prove the other properties of parallel lines with sufficient clearness.

This principle in fact, is that having drawn through the point P (*fig.* 15.) the line L N parallel to A B, another straight line, such as Q O, through the same point P cannot be parallel to A B; that is to say, that if such a line be continued to a sufficient distance, it must ultimately meet the line A B.

We shall presently show that perpendiculars drawn from every point of the line L N to the line A B are equal, and that therefore every point of the line L N is at the same distance from the line A B. Now it will be evident that no other line through P can enjoy this property. The line Q O, on the right of the point P, will have its points at a less distance from A B than P M, and on the left of the point P it will have its points at a greater distance than P M; indeed it is sufficiently apparent, that P O continually approaches the line A B, and P Q continually recedes from it, and that if the line Q O be continued to a sufficient distance to the right, it must at length meet the line A B, if the latter be also continued in the same direction.

(31.) It will now be easy to show that if two parallel lines, A B and L N, be drawn, any line whatever, such as P M, which is perpendicular to one of those parallel lines, must be also perpendicular to the other. Let us suppose that P M is perpendicular to A B, it must then be also perpendicular to L N; for if it were not, let another line Q O be drawn through P at right angles to P M. That line Q O would then, according to what has been already proved (29.), be also parallel to A B, and we should thus have two different lines passing through the point P, both parallel to A B. This has been assumed to be impossible, and therefore the line P M must be perpendicular to L N, as well as to A B; and in general every line which is perpendicular to one of two parallel lines must be also perpendicular to the other.

(32.) It is evident that all the angles marked *a* in *fig.* 16. will be right angles, supposing the transverse line to cross either of the parallels perpendicularly.

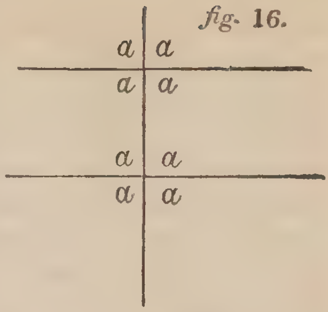


fig. 16.

(33.) It is evident, also, that if a straight line be perpendicular to any one of several parallels, as in *fig.* 17., it will be perpendicular to all of them.

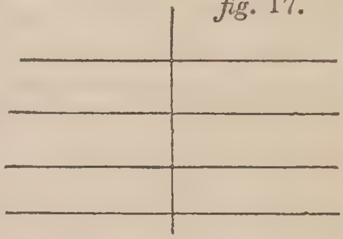


fig. 17.

(34.) Two parallel lines are every where equally distant, or, in other words, perpendiculars drawn from every point in either to the other are equal.

Let A B and L N, (*fig.* 18.) be two parallel lines.

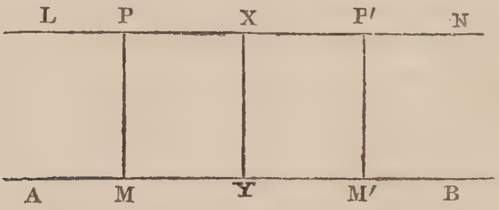


fig. 18.

If from any two points P P′, perpendiculars, P M and P′ M′, be drawn to the other, they will be equal. It has been already proved (31.) that P M and P′ M′ must be perpendicular to L N, as well as to A B, and therefore the angles at the four points P M P′ M′ are all right angles, and are therefore equal. Now let the point X be taken midway between P and P′ and let the line X Y be drawn perpendicular to both parallels; let the paper on the left of X Y be conceived to be folded over so as to cover the paper to the right, the fold being made to correspond with the line X Y. The line X P must in this case coincide with the line X P′, because of the equality of the two right angles at X; and the point P must fall upon the point P′, because of the equality of the distances X P and X P′, also the line P M must fall upon

the line P′ M′, because of the equality of the right angles at P and P′; also the line Y A must fall upon the line Y B, because of the equality of the right angles at Y. Since then Y A falls upon Y B, and P M upon P′M′, the point M must fall upon the point M′. The perpendicular P M must therefore precisely cover the perpendicular P′ M′, and therefore these perpendiculars must be equal; and in the same manner it may be shown that all lines drawn from one of two parallels perpendicular to the other must be equal.

(35.) Lines which are perpendicular to parallel lines will themselves be parallel; for it has been already proved that lines which form right angles with the same line must be parallel.

(36.) Hence it follows, also, that the parts of parallel lines included between perpendiculars to them must be equal. Thus in *fig.* 18. the distance P P′ is equal to the distance M M′.

(37.) If two systems, each consisting of several parallel lines, cross each other at right angles, all the parts of one system included between any two lines of the other system will be equal.

The ordinary framing of a window consists of two systems of lines of this kind; also the shelves and upright standards of bookcases, the panelling of doors and presses, and various other structures produced in carpentry, afford similar examples.

All fabrics produced in the loom, consist of two systems of parallel threads, crossing each other at right angles; so interlaced, however, as to give strength and consistency to the cloth.

A railway consists of two or more parallel lines of iron bars, called *rails*, which are supported upon props. The wheels of the carriages are fixed upon axles, so that their distance asunder shall correspond precisely with the length of perpendicular lines drawn between the parallel rails. As the axle of the wheel moves with the carriage in a direction parallel to the rails, it will always remain perpendicular to them. Since, there-

fore, it takes successively positions of this kind, similar to the positions of P M and P′ M′ with reference to the parallels A B and L N (*fig.* 18.), it follows that the wheels must always move over equal lengths of the rails in the same time.

(38.) The parallelism of lines which are perpendicular to the same line, is the principle on which the application of the instrument called the T square depends. This is an instrument which consists of two straight and flat rulers fixed at right angles to each other, as represented in *fig.* 19. A straight line being drawn in a direction perpendicular to that in which it is required to draw the parallels, the cross piece of the T ruler is laid upon this line, and the piece at right angles to this gives the direction of one of the parallels; the ruler being moved along the paper, keeping the cross-piece coincident with the line first described, any number of parallel lines may be drawn.

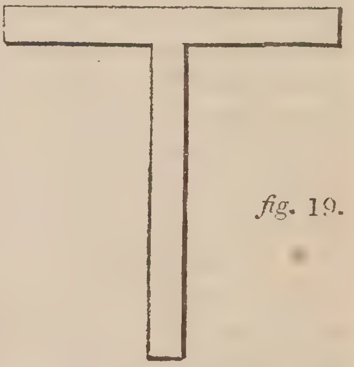


fig. 19.

The uniformity of distance which characterises parallel lines, is the principle upon which numerous instruments and processes in the arts are founded.

(39.) The rolling parallel ruler is an instrument by which any number of lines may be drawn parallel to a given line, and at any required distances from each other. This ruler consists of a flat piece of wood with a straight edge, usually divided into inches and parts of an inch. In the ruler, near its extremities at A and B (*fig.* 20.), are inserted two rollers, by which

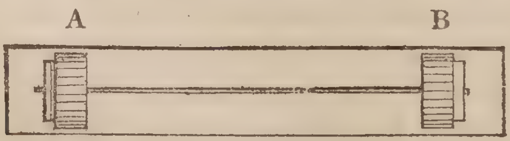


fig. 20.

the ruler is capable of moving at right angles to the

direction of its edge. These rollers are fixed upon the same axis which extends along the ruler parallel to its edge. If the circumferences of these rollers measure an inch, they may be divided into parts of an inch, so that the space through which the ruler is moved as they turn may be accurately observed. This space will be the distance between lines whose directions are determined by the edge of the ruler in different positions.

The characters in music consist of dots placed upon or between a system of five parallel lines at equal distances from each other. These lines are sometimes drawn upon paper by an instrument called a *music pen*, consisting of five points at distances corresponding to the distances between the lines; such an instrument is merely a contrivance for drawing one particular system of equidistant lines.

The same principle is more extensively applied in the mechanism used in ruling paper, where a number of points supplied with ink are maintained at fixed distances from each other, and are either moved over the paper on which the lines are required to be traced, or held in contact with the paper while the latter is moved under them.

The uniformity and precision with which thread is produced in the modern spinning frames, depends upon the same principle. Two frames, one of which is fixed and the other moveable, are placed parallel to each other, one supporting as many bobbins as there are threads to be simultaneously spun, and the other supporting a corresponding number of spindles. While the threads receive the rotatory motion which twists them, the one frame is moved from the other on a railway by which its parallelism to the latter is preserved; during the motion the threads are extended between the moving and the fixed frames in directions at right angles to these. Under such circumstances, it must be evident that the threads will be all equally stretched; and as the same number of revolutions are at

the same time imparted to all the spindles, all the threads will be equally twisted.

(40.) It has been already shown, that if two straight lines form right angles with a third line, they will be parallel; but it may also be shown that this principle is still more general, inasmuch as two straight lines which are inclined at equal angles to a third, whether those angles be right or not, will be parallel. Let A B and L N (*fig.* 21.) be crossed by the line P M, and

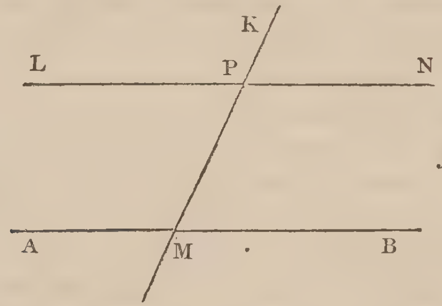
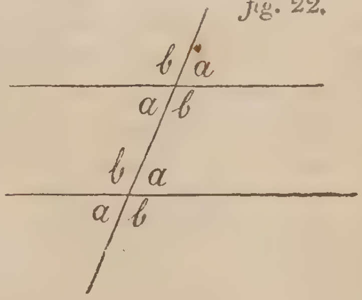


fig. 21.

let the angle K P N be equal to the angle P M B, then the lines A B and L N must be parallel; for if they were not parallel, they would meet at some point more or less remote. And the lines P N and M B, being at different distances from the perpendicular drawn from the point where these lines would meet to the line M P K, must necessarily make unequal angles with the line M P K, that which is more remote from the perpendicular being more inclined than that which is nearer to it. (24). Therefore the lines L N and A B cannot meet, and therefore must be parallel.

(41.) From what has been just proved, combined with the fact that angles vertically opposite will be equal (20.), it follows, that when two parallel lines are crossed obliquely by a third line, as in *fig.* 22., the angles which are marked by the same letters in this figure will be equal; and it is also obvi-

fig. 22.

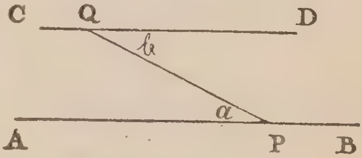


ous that the angles marked b are the supplements of the angles marked a.

(42.) When a line joins two parallel lines, angles placed on contrary sides of it, such as the angles marked a and b (*fig.* 23.) are called *alternate angles;* and from what has been already shown, it appears, that when a line joins parallel lines the alternate angles will be equal.

(43.) By this property of parallels, a line may be drawn parallel to a given line if we are furnished with the pattern or model of any angle whatever. Let it be required to draw through the point Q (*fig.* 23.) a line parallel to A B; from Q draw a line Q P, making with the line A B an angle a, of which we possess a model or pattern; with the same model draw a line Q D, making the angle b equal to the angle a. The line Q D will then be parallel to A B, since the alternate angles are equal.

fig. 23.



The property by which parallel lines are equidistant, and have equal parts included between perpendiculars to them, is of extensive use in mechanics. It is by virtue of this property that when a progressive motion is imparted to a body all its parts move in parallel lines, preserving the same relative position amongst each other. This motion is sometimes, in the arts, called a *parallel motion;* and it is frequently of importance to produce such a motion with the last degree of mechanical precision. The piston of a steam-engine, and the rod which it drives, receive such a motion; and any deviation from it would be attended with consequences injurious to the machinery. The whole mass of the piston and its rod must be moved, so that every point of it shall describe lines exactly parallel to the direction of the cylinder.

CHAP. IV.

ON TRIANGLES.

(44.) If three points, which are not in the direction of the same straight line, be joined by three straight lines, these three straight lines will include a space, and a geometrical figure will be formed, called a *triangle* from the circumstance of its having three angles: the three straight lines which enclose the figure are called the *sides* of the triangle.

(45.) When a triangle is drawn with one of its sides horizontal, it is customary to distinguish that side from the others by calling it the *base*.

The triangle is a figure of great importance in geometrical inquiries, because all figures bounded by straight lines are capable of being resolved into triangles, and of having their properties investigated by, and derived from, the properties of triangles.

(46.) In investigating and comparing triangles, there are seven quantities or magnitudes which will demand attention in each triangle, viz. the three sides, the three angles, and the quantity of superficial space included within the sides.

(47.) Among the various relations which subsist between these several quantities connected with triangles, the most important and remarkable is one, which respects the three angles. In every triangle, whatever be its magnitude or form, the three angles, when added together, always amount to precisely 180°. We shall hereafter show that this is only a particular case of a much more general geometrical principle; but, meanwhile, we shall present it in its restricted form.

Through the vertex of any angle *a* of a triangle, *fig.* 24., let a line M N be drawn parallel to the opposite

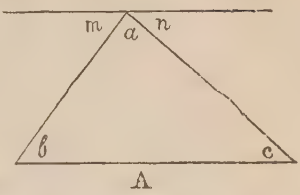


fig. 24.

side A. By what has been already proved of the properties of parallel lines, the sides of the triangle will be inclined to M N at the same angles as those at which they are inclined to its parallel A; that is to say, the angle m is equal to the angle b, and the angle n is equal to the angle c (42.). Thus the three angles, m, a, n, are equal to the three angles of the triangle; but since M N is a straight line, these three angles, m, a, n, must make up 180°; therefore the angles of the triangle, if added together, would likewise make up 180°.

(48.) If by any change in the position or magnitude of the sides two angles of a triangle are varied in magnitude while the remaining angle remains unchanged, one of the varying angles must increase and the other decrease by exactly the same amount: this immediately follows from the principle just established that the sum of the three angles is unalterable.

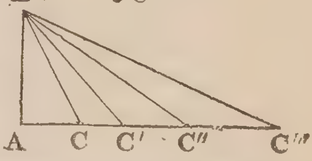


fig. 25.

In *fig.* 25. let the angle A B C be supposed to be gradually increased by moving the side B C on the point B, as a pivot, the successive positions which the side B C would take, as the angle A B C is increased by this motion, are represented in the figure. It is evident then that as C recedes from A, the angle at C gradually diminishes, and its decrements from one position to another must be equal to the corresponding increments of the angle A B C.

(49.) Hence it follows, that the angle A C B exceeds the angle A C′ B by the magnitude of the angle C B C′. This is generally enounced as a distinct proposition, in the following terms, considering C B C′ as an independent triangle, and A C B its external angle: —

*In any triangle, if one of the sides be produced**,

* "To produce" is the techinal term used in geometry, to signify extending or prolonging a straight line.

the external angle which will be formed will be equal to the two remote internal angles taken together.

(50.) In general, by the external angle of any figure is meant an angle which is formed by producing one of the sides through the vertex of one of the angles. In *fig.* 26. all the sides of the figure are thus produced; and adjacent to every internal angle there is a corresponding external angle; and it is evident that each external angle is the supplement of the adjacent internal angle.

fig. 26.

(51.) The following consequences flow obviously from the principle, that the sum of the three angles of a triangle is equal to 180° : —

(52.) If one angle of a triangle is right, the sum of the other two is equal to a right angle.

(53.) If one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle

(54.) An obtuse angle of a triangle is greater, and an acute angle less, than the sum of the other two angles.

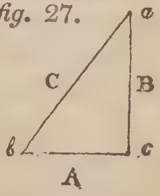
(55.) If one angle of a triangle be greater than the sum of the other two, it must be obtuse; and if it be less than the sum of the other two, it must be acute.

(56.) If two angles of a triangle be known, the remaining angle may be found by subtracting the sum of the two known angles from 180°.

(57.) If two triangles have two angles in the one equal to two angles in the other, the remaining angles must be equal.

(58.) A triangle cannot have more than one angle right or obtuse, and consequently every triangle must have at least two acute angles.

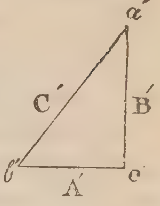
fig. 27.



(59.) In *fig.* 27. is represented a triangle, whose sides are expressed by the letters A, B and C, and whose angles are expressed by the letters *a*, *b* and *c*.

In *fig.* 28. is represented another tri-

fig. 28.



angle, whose sides are expressed by the letters A′, B′ and C′, and whose angles are expressed by the letters a', b' and c'.

If the sides B and C, in *fig.* 27., be respectively equal to the sides B′ and C′, in *fig.* 28., and the angle a be equal to the angle a', then the remaining side A will be equal to the remaining side A′, and the angles b and c will be respectively equal to the angles b' and c'. The superficial dimensions of the triangles will also be equal, and the figures will be so precisely alike, that the one may be placed upon the other, so as exactly to cover it. To prove this, let us imagine that a pattern of the triangle in *fig.* 28. is executed in card; let the vertex of the angle a' in the pattern be placed upon the vertex of the angle a, and let the side C′ be laid upon the side C, and finally, let the pattern of *fig.* 28. be turned down upon *fig.* 27.; since the side C′ coincides with the side C, and the angle a' is equal to the angle a, the side B′ of the pattern must be upon the side B. Also, since the sides C and C′ are equal, and also the sides B and B′, the ends of these sides must coincide respectively; that is, the vertex of the angle b' of the pattern must lie upon the vertex of the angle b, and the vertex of the angle c' of the pattern must lie upon the vertex of the angle c; the ends of the side A′ of the pattern will therefore coincide with the ends of the side A, and consequently these sides must lie one upon the other. The pattern, therefore, of *fig.* 28. will precisely cover *fig.* 27., and the angle b' will be equal to the angle b, the angle c' will be equal to the angle c, and the superficial dimensions of the triangles will be the same.

In fact, the triangles are in this case in all respects equal and similar.

This important truth is usually enounced in geometry, in the following terms: —

Two triangles, having two sides in the one equal to two sides in the other each to each, and the angles included between these sides equal, will have the remaining sides

equal — the remaining angles equal each to each, and their areas equal.

(60.) The term *area* is used in geometry to express the superficial dimensions of any figure.

(61.) In the two triangles, expressed in *figs.* 29, 30., let it be granted that the sides marked C and C′ are equal, that the angle *b* is equal to the angle *b*′, and the angle *a* equal to the angle *a*′; under these circumstances it may be proved, that the remaining sides and angles of the triangles will be equal each to each, and that their superficial dimensions will be equal.

fig. 29.

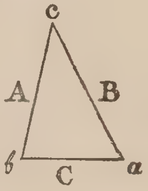
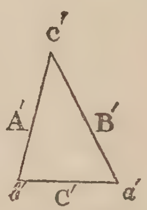


fig. 30.



As before, let a pattern of *fig.* 30. be executed in card, and let the vertex of the angle *b*′ be placed upon the vertex of the angle *b*, and the side C′ be placed upon the side C; then, because of the equality of these sides, the vertex of the angle *a*′ will fall upon the vertex of the angle *a*. Let the pattern be laid over the triangle, *fig.* 29.; and since the angle *b*′ is equal to the angle *b*, the side A′ must fall upon the side A; and, in like manner, since the angle *a*′ is equal to the angle *a*, the side B′ must fall upon the side B. And since the sides A′ and B′ fall respectively upon the sides A and B, the vertex of the angle *c*′ must fall upon the vertex of the angle *c*; and therefore the angle *c*′ must be equal to the angle *c*, and the triangles must in all respects be mutually coincident and equal: the side A′ being equal to the side A, the side B′ to the side B, and the superficial dimensions being the same.

This proposition is usually enounced in geometry, in the following manner: —

If two triangles have a side in the one equal to a side in the other, and the angles between which that side is placed equal each to each, then the remaining sides and angles will be equal each to each, and the areas of the triangles will be equal.

By a process similar to the above, it may be demonstrated that if the angles c and b be equal to the angles c' and b' respectively, the sides C and C′ being at the same time equal, the triangles will admit of superposition; and will be therefore in all respects equal.

This proposition is usually enounced in geometry as follows: —

If two triangles have a side in the one equal to a side in the other, and two angles similarly placed with regard to these two sides equal, the triangles will be in all respects equal.

The proposition, thus enounced, also comprehends the former one.

(62.) A triangle differs from all other rectilinear figures in this, that if its sides be united at the angles by pivots or hinges, it will nevertheless be incapable of having its form altered, and the pivots or hinges can have no play.

This would evidently not be the case with figures having a greater number of sides. If the four-sided figure represented, in *fig.* 31., had its sides united at the angles by pivots, it might be obviously converted by merely turning the sides upon their joints into the figure represented in *fig.* 32., and it might receive an unlimited variety of other forms, all compatible with the unaltered lengths of the sides — and the same would be true of any other figure having more than three sides; but in a triangle, any attempt to cause one of the sides to move upon the pivot at one of the angles, is resisted by its connection with the other sides, with which connection any such motion is incompatible.

fig. 31.

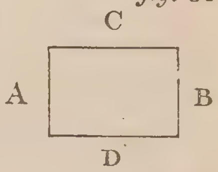
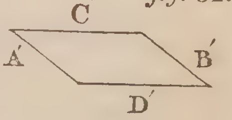


fig. 32.



It is evident from this fact, that if two triangles have their three sides respectively equal, their angles must also be equal, and they must admit of superposition so as exactly to cover one another, otherwise it would follow

that with the same sides a triangle would admit of two different forms.

This proposition is usually enounced thus: —

If two triangles have the three sides of one equal to the three sides of the other each to each, then the three angles will be equal each to each, and their areas will be equal.

(63.) When two sides of a triangle are equal to each other, it is called an *isosceles triangle*, and in that case the remaining side is usually called the base. In *fig.* 33., if the sides A and B be equal, the angles a and b opposite these sides will also be equal; and, on the other hand, if the angles a and b be equal, the sides A and B opposite them will also be equal.

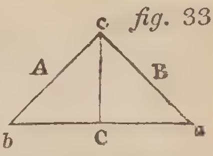


fig. 33.

For if a line be drawn from the vertex of the angle c to the middle point of the base, it will divide the whole triangle into two triangles, whose sides will be respectively equal, and therefore whose angles will be equal: hence the angle a will be equal to the angle b.

If, on the other hand, it be granted that the angle a is equal to the angle b, let a line be conceived to be drawn from the vertex of the angle c, dividing that angle into two equal parts; this line will thus resolve the proposed triangle into two, having a side common, and two angles respectively equal: therefore the side A will be equal to the side B.

(64.) The line c C, which joins the vertex of an isosceles triangle with the middle point of the base, is perpendicular to the base, since the angles at each side of it have been proved equal; and it also bisects the vertical angle c, or divides it into two equal angles.

For if the triangle be conceived to be folded over, so that the part of it on the right of the line c C shall fall upon the part on the left of that line, these parts will exactly cover each other.

(65.) A line which divides any figure in this manner is said to divide it *symmetrically*.

(66.) If a perpendicular c C, drawn from the vertex of

a triangle to the base bisect either the base or the vertical angle, the triangle will be isosceles.

For if it bisect the base, let the part of the triangle to the left of *c* C be folded over that part to the right, since the angles at C are equal, the part of the base to the left of C will fall upon the part to the right; and since these parts are equal, the vertex of the angle *b* will fall upon the vertex of the angle *a*, and the side A will coincide with the side B, and will therefore be equal to it.

If the perpendicular to the base bisect the angle *c*; then doubling over the part to the left of *c* C upon the part to the right, the side A will fall upon the side B, because the angle at C is bisected by the perpendicular, and the part of the base to the left of C will fall upon the part of the base to the right of C, because the angles at C are equal; therefore the vertex of the angle *b* will fall upon the vertex of the angle *a*, and the side A will fall upon the side B, and will be equal to it.

(67.) These properties furnish the means of solving the problem to bisect an angle.

If *c* be the angle to be bisected, take equal parts A and B upon its sides, and draw a base C, so as to form an isosceles triangle; from the vertex of the angle *c* draw a line at right angles to this base, which may be done by a square; this line will, by what has already been proved, bisect the given angle.

(68.) The same principles furnish a solution of the problem to bisect a given straight line.

If the base C (*fig.* 33.) be the proposed straight line which is to be bisected, draw at its extremities any two equal acute angles, which may be done by the pattern of one acute angle, the sides of these acute angles will form an isosceles triangle (63); and if the perpendicular be drawn from the vertex *c* to the base of this isosceles triangle, that perpendicular will bisect the base (65.).

(69.) If the vertical angle, *c*, of an isosceles triangle were right, the base angles, *a*, *b*, would be each 45°, since all the three angles must be equal to 180° (47.).

(70.) The angles at the base of an isosceles triangle must always be acute, since they are equal, and since more than one right or obtuse angle cannot exist in the same triangle (58.).

(71.) A triangle having three equal sides is called an *equilateral triangle* (*fig.* 34.).

fig. 34.

An equilateral triangle may be regarded as an isosceles triangle, any one of the three sides being taken as base; and as it has been proved that the angles at the base of an isosceles triangle are equal (63.), it follows that the three angles of an equilateral triangle are equal.

(72.) Also, if the three angles of any triangle are equal, the three sides must be equal, because it will be an isosceles triangle, according to what has already been proved, in whatever position it may be placed (63.).

Thus an equilateral triangle is equiangular, and an equiangular triangle is equilateral.

Since the three angles are together equal to 180°, each angle of an equilateral triangle must be 60°, or two thirds of a right angle.

The equilateral triangle presents the first example in geometry of a *symmetrical figure*.

Since a perpendicular from the vertex of an isosceles triangle upon the base divides it symmetrically (64.), an equilateral triangle will be divided symmetrically by a perpendicular from the vertex of any angle on the opposite side.

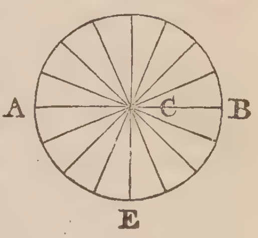
The isosceles triangle is extensively used in architecture and in carpentry. It is the form usually given to the roofs of buildings, and to the pediment which surmounts and adorns porticos, doors, and windows. In the Greek architecture, the character of the isosceles is obtuse; in the Gothic, acute.

CHAP. V.

OF CIRCLES.

(73.) If a straight line have one of its extremities placed at a fixed point, C (*fig.* 35.), and be made to revolve round that point as a pivot, the other extremity will trace a line, every point of which will be equally distant from the point C. Such a line is called *a circle*, the point C is called its *centre*, and the line C B its *radius*; the space inclosed within the curve is called the *area* of the circle, and the curved line itself is called the *circumference* of the circle.

fig. 35.



(74.) A straight line extending across the circle, through its centre, and terminated in its circumference, is called a *diameter*.

A diameter consists evidently of two radii placed in the same straight line, and it is therefore equal to twice the radius of the circle; all diameters are therefore equal to each other.

The art of turning consists in the production of this figure by mechanical means. The substance on which the circular form is required to be conferred is placed in a machine called a *lathe*, which gives it a motion of rotation round a certain point as a centre; the edge or point of a cutting tool is placed at a distance from this centre, equal to the radius of the circle which it is desired to form; as the substance revolves, the edge or point removes every part of it which is more distant from the centre than the proposed radius, and consequently the circular form is given to what remains.

(75.) The circle is a perfectly symmetrical figure; for if it be made to revolve round its own centre no change whatever will take place consequent on the change of position of the parts: every part of its circumference being at the same distance from the centre, each point as it revolves takes the place of the preceding point, and no new portion of space is either vacated or occupied during this motion. The circle is unique in this property, which is possessed by no other figure whatever.

It is in virtue of this property that the axles of wheels, shafts, and other solids which are required to revolve within a hollow mould or casing of their own form, must be circular. If they were of any other form, when placed in the mould or casing they would be incapable of revolving without carrying the mould or casing round with them.

Wheels, which are intended to maintain a carriage supported by them always at the same height above the road on which they roll, must necessarily be circles, with the axle of the wheel in their centre. The distance of the centre of the axle from the road will be equal to the distance of the centre of the wheel from its edge. In the circle, this distance is always the same, and it is the only figure which has a point within it possessing this property.

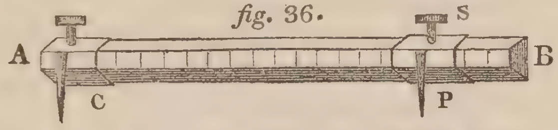
(76.) The instruments by which circles are most commonly described are called *compasses*, and consist of two straight and equal legs connected together at one end by a joint, on which they are capable of moving, and terminating at the other ends in points, one of which carries a pen or pencil; the point of one leg is placed at the centre of the circle which it is intended to describe, while the other leg, carrying the pen or pencil, is made to revolve round, pressing the pen or pencil on the paper intended to receive the trace of the circumference.

When it is required to describe a circle with a radius too great for the space of the compasses, it may be done by attaching a piece of string with a pin to the

proposed centre, and looping into the string a pen or pencil at the proper distance for the required circle.

(77.) An instrument called a *beam compass* is also intended for describing circles of greater radius than those to which ordinary compasses can be conveniently applied. The beam compass consists of a straight bar A B (*fig.* 36.) usually divided into inches and parts of

fig. 36.



an inch. At the commencement of the divisions there is a steel point C fixed projecting from the lower face of the bar. This point is intended to mark the centre of the circle to be described. A brass slider S is placed upon the bar, furnished with a clamping screw to fix its position at any required distance from the point C, which slider carries a point or pencil P, projecting downwards from the lower side of the slider.

In the application of the instrument to describe circles, the slider is moved along the bar until the distance of the describing point P, from the central point C, shall be equal to the radius of the required circle. The sliding piece is then fixed in its position by the clamping screw, and the central point C being placed at the centre of the proposed circle, the bar is moved round, the describing point P being pressed upon the paper so as to leave the trace of the circumference of the required circle.

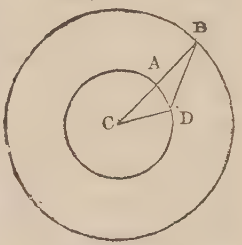
(78.) If two circles have equal radii they will be equal in every other respect; for if the centre of the one be imagined to be placed on the centre of the other, the circumference of the one must coincide in every point with the circumference of the other, since every part of the circumference of each will be at the same distance from their common centre.

(79.) If two circles with different radii, be drawn round the same centre, every part of the circumference of one will be at the same distance from the circum-

ference of the other; that distance being measured in the direction of their common centre. It is evident that this distance will be the difference between the radii of the two circles, and will be the least distance between their circumferences.

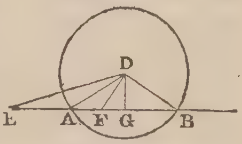
(80.) If two circles with unequal radii be described round the same centre (*fig.* 37.), any distance between them, such as A B, drawn in such a direction that if it be produced inwards it will pass through the centre, will be less than any other distance, such as B D. To prove this, it is only necessary to observe, that the distance from B to C round the angle D is greater than the direct distance B A C. If from these two distances the equal lines C D and C A be taken away, the remainder B D in the one case will continue to be greater than the remainder B A in the other case.

fig. 37.



(81.) If a straight line be drawn joining any two points A and B (*fig.* 38.) in the circumference of a circle, every part of that straight line must be within the circle; and if the same straight line be continued beyond the points A and B on either side, every other part of it must fall outside the circle.

fig. 38.



For if D be the centre of the circle, let the perpendicular D G to the line A B be drawn, and also let a line D F be drawn to any other point between A and B, and a line D E to any point beyond A and B. The line D G, being perpendicular to A B, is shorter than D A the radius of the circle, and therefore the point G is within the circle. Also the line D F, being nearer to the perpendicular than D A, will be less than D A (24.), and being less than the radius, the point F must be within the circle; and the same observation may be applied to every point of the line between A and B. On the other hand, the line D E, being more distant from

the perpendicular than D A, is greater than the radius (24.), and therefore the point E is outside the circle; and the same observations may be applied to every point of the line beyond A and B.

(82.) Hence it follows, that a straight line cannot meet the circumference of a circle in more than two points, because every point of the line between these points will be within the circumference, and every other point will be without it.

(83.) If a straight line be drawn through any point B, *fig.* 39., on the circumference of a circle in a direction perpendicular to the radius B C, every point of that straight line on either side of the point B will lie outside the circle; for let a line be drawn from the centre C to any point, such as I, on the straight line at either side of B, this line C I will be longer than the perpendicular C B (21.), and therefore the distance of I from the centre of the circle will be greater than the radius, and therefore the point I will be outside the circle; and the same observations will be applicable to every point upon the line F B F, except the point B.

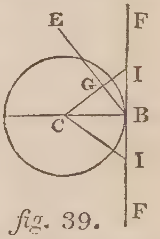


fig. 39.

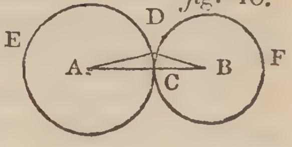
(84.) A straight line, such as F B F, which meets a circle at one point B, and lies altogether outside the circle, is said to *touch* the circle at B, and is called a *tangent*.

(85.) Any straight line drawn from B, such as B E, if it be not perpendicular to B C, must pass within the circle on that side at which it makes an acute angle with B C. For if the line C G be drawn perpendicular to B E, C G will be less than C B (21), and the distance of the point G from the centre, being less than the radius of the circle, the point G must be within the circle.

(86.) A straight line, which lies partly within and partly without a circle, is said to intersect or cut the circle, and is called a *secant*.

(87.) If the distance between the centres A and B (*fig.* 40.) of two circles be equal to the sum of their radii, the circumferences of these circles will meet at one, and only one point C, and will be altogether outside each other.

fig. 40.



For if a part A C be taken upon A B equal to the radius of the circle A, the remainder B C must be equal to the radius of the other circle. Since the point C therefore is, at distances from the two centres, equal to the radii of the circles respectively, it must be on the circumference of both circles; that is to say, the circumferences of both circles must pass through the point C. But if any other point, such as D, be taken on the circumference of the circle A, the distance of that point D from the centre of the other circle B will be greater than B C. This may be easily shown, for the distance of B from A, measured round the angle D, will be greater than the direct distance of B from A by C. If, then, from both of these the distances of D and C from A be taken away, the remainder B D will be greater than the remainder B C. The distance therefore of D from B is greater than the radius of the circle B, and therefore the point D must be outside the circle B; and the same will be true of any point in the circumference of the circle A.

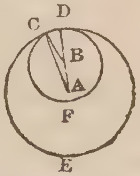
In the same manner it may be shown, that every point of the circumference of the circle B, except the point C, lies outside the circle A.

Two circles, situate with respect to each other in in this way, are said to *touch externally.* The condition therefore of the *external contact* of circles is, that the distance between their centres should be equal to the sum of their radii; and it follows obviously, from what has been just explained, that the straight line joining the centres of circles which touch externally, must pass through their point of contact.

(88.) If the distance between the centres A and B,

fig. 41., of two circles be equal to the difference between their radii, the circumferences of these circles will meet each other in one and only one point D, and every other point of the lesser circle will be within the circumference of the greater circle.

fig. 41.



For if the line AB be continued until A D be equal to the radius of the greater circle, then B D must be equal to the radius of the lesser, since A B is the difference of the radii. Therefore, since the point D is at distances from the two centres respectively equal to the radii, the two circumferences must pass through that point. But if from B a line B C be drawn to any point in the circumference of the lesser circle, and another line from A to the same point C, the distance A C will be less than the distance A B C, and therefore less than the distance A B D; therefore the distance of C, on the circumference of the lesser circle from the centre of the greater circle, will be less than the radius of the latter, and consequently the point C must be *within* the circumference of the greater circle; and in the same manner it may be shown that every point of the circumference of the lesser circle, except the point D, will be within the greater circle.

(89.) Two circles, placed in the manner here described, are said to *touch internally.*

(90.) The condition of *internal contact* is, therefore, that the line joining the centres shall be equal to the difference of the radii.

(91.) It is evident from what has been explained, that if the distance between the centres of two circles be equal to the difference of their radii, the straight line joining their centres, will, if produced, pass through their point of contact.

(92.) It also follows (83.), that if a straight line be drawn through the point of contact of two circles which touch each other, whether internally or externally, perpendicular to the line joining their centres, that straight

line will be a tangent to both circles (*fig.* 42.).

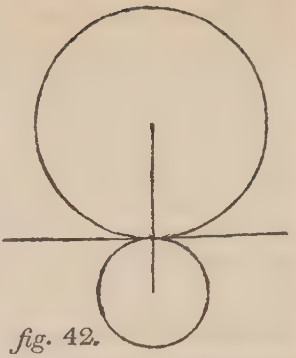


fig. 42.

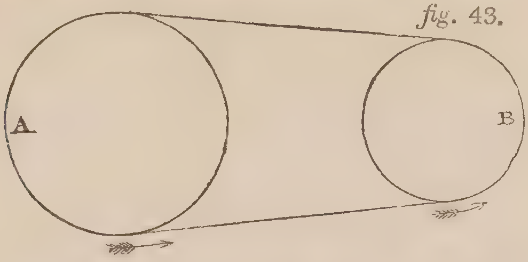
The properties of circles touching each other, and touching straight lines, are of extensive use in the arts; the circles which form wheels in machinery, are made to act upon one another by their surfaces being brought into contact. The distance between the centres or axles of the wheels in this case, if the wheels be outside each other, must be equal to the sum of the radii of the wheels. When one circle is made to revolve round its axle, it must either slide upon the other circle, or compel the other to turn with it. The sliding is sometimes resisted by the roughness produced on the edges of the circles of the two wheels which are thus in close contact with each other. This roughness is produced by forming the edges of the wheels of wood with its grain placed in contrary directions, or by facing the edges of the wheels with leather; but the action of the wheels upon each other is most commonly effected by forming teeth on the edges of each wheel, of the same magnitude and with the same intervals between them: the teeth of one wheel inserting themselves between the teeth of the other, one cannot revolve without causing the other to revolve at the same time.

The contact of a straight line with a circle is also frequently used in the arts. The most common example of this is, when a strap or band is carried round a part of the circumference of a wheel, and extending to a distance is carried round the circumference of another wheel, sufficient tension being given to it to produce such a degree of friction or adhesion between it and the wheel, that the wheel cannot revolve without moving the strap with it.

In this manner the motion of one wheel may be conveyed to another at a distance from it. If both wheels are intended to revolve in the same direction, the strap

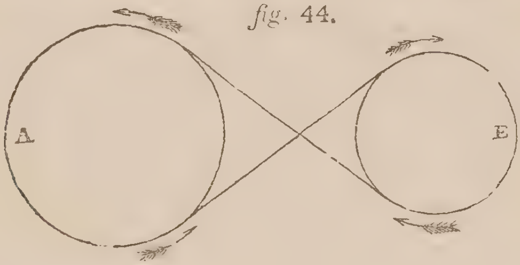
will connect them, as represented in *fig.* 43. But if

fig. 43.



they are required to revolve in contrary directions then the strap must be crossed between them, as in *fig.* 44.

fig. 44.



Sometimes a third wheel or roller (D) is introduced, capable of being shifted in its position so as to vary the tension of the strap *fig.* 45.

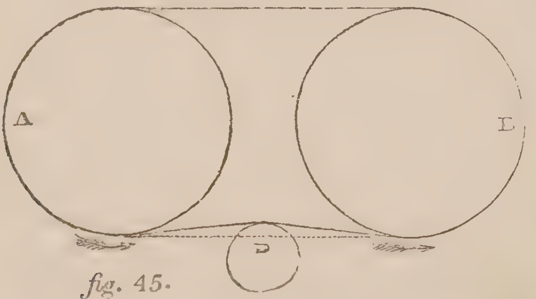


fig. 45.

In the application of wheels to carriages, the line of road is usually a tangent to the circle of the wheel; the point of contact being the point where the weight of the carriage presses upon the road.

The axle in practice never precisely corresponds in size with the nave or box in which it turns, the latter being always a little larger. It is evident, that under such cir-

cumstances, the axle and the nave are circles which touch each other internally, the point of contact being the point where the axle rests upon the nave.

When rollers are applied to shift the position of heavy weights, the platform which the rollers support and the road on which they rest are both tangents to the circles of the rollers.

(93.) If two equal angles be formed by radii diverging from the centre of the same circle, the arcs included between such radii will be equal; for if the sides of one angle be conceived to be applied to the sides of the other, they will coincide in consequence of the equality of the angles; and every part of the arcs must coincide, since they will be at the same distances from the centre.

(94.) Hence if the space round the centre of a circle be divided into any number of equal angles, the circumference will be divided into a corresponding number of equal arcs.

(95.) Two diameters of a circle, which cross each other at right angles, will divide the circumference into four equal parts called *quadrants*, and any two radii at right angles to each other will include between them a fourth part of the circumference.

(96.) In general it will be perceived that angles and circular arcs may be taken as the measures of each other, and the subdivision of angles into degrees, already explained, will be equally applicable to arcs. The circumference of a circle therefore will consist of 360° and the quadrant of 90°.

(97.) The subdivision of the circle is carried further, each degree, whether of angles or arcs, being supposed to be divided into sixty equal parts called *minutes*, and each minute again into sixty equal parts called *seconds*.

This system of division is sometimes carried even further, a second being divided into sixty equal parts called *thirds*; but it is more usual to express small angles or arcs in decimal parts of a second.

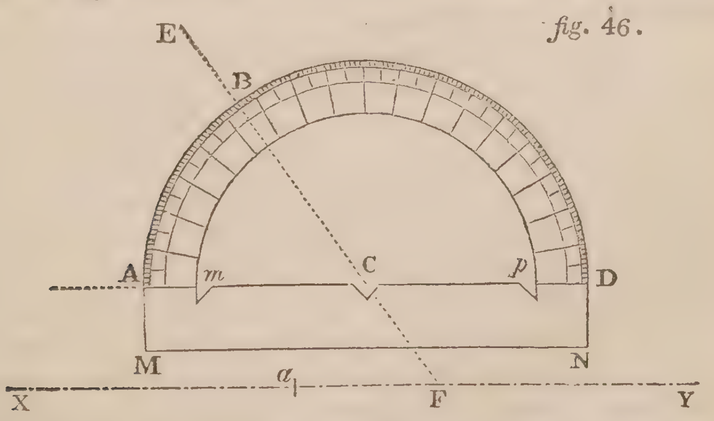
(98.) The circumference of the earth, considered as a circle, is subdivided in this way; one degree measuring

60 geographical miles, and the circumference of the earth therefore consisting of 360°, and measuring 21,600 miles. One minute of the earth's surface will therefore correspond to one geographical mile.

Instruments for measuring angles are founded upon the principle that arcs are proportional to angles. Such instruments usually consist of either a part of a circle, or an entire circle of brass or metal, on the surface of which is accurately engraven its divisions, in conformity with the system of degrees, minutes, and seconds already explained. Such instruments are usually furnished with a moveable radius; and in the measurement of angles the fixed radius, which passes through the first division of the scale, is directed along one side of the angle to be measured, and the moveable radius is shifted in its position until it is directed along the other side. The angle between the two radii is then indicated by the magnitude of the graduated arc of the circular limb of the instrument between them.

An instrument called a protractor is used in mechanical and geometrical drawing for measuring angles, and for laying down on paper angles of any required magnitude. This instrument consists of a brass semicircle A B D, *fig.* 46., the circumference of which is divided

fig. 46.



into degrees and parts of a degree. The ends of the semicircle are connected by a flat plate of brass A D, the sides of which are perfectly straight and parallel: the

inner side being the diameter of the semicircle, the metal is cut from the space between the graduated arch and the diameter. The points of the angular incisions, marked m, C, p, correspond precisely to the extremities of the diameter and the centre of the semicircular arc.

In the application of this instrument, let us suppose, for example, that it is required to draw from the point a a straight line, making an angle of any required magnitude with a given straight line (E F). Let the centre C of the protractor be placed any where upon the line E F, and taking the point B on the protractor, so that the distance from A to B on the graduated semicircle shall correspond with the magnitude of the required angle, let the protractor be placed so that the point B shall also lie upon the straight line E F: let the protractor be now moved towards the point a, keeping the points B and C on the straight line E F until the edge M N of the diametral bar of the protractor shall pass through a; let a line (X Y) then be drawn, using that edge as a ruler: such a line will form with the line E F the required angle; for since the edge M N is parallel to the diameter, the line X Y must make with E F the same angle as the diameter forms with it, and the latter angle is obviously measured by the arch A B, and is therefore the required angle.

(99.) The division of the circumference of a circle into any required number of equal parts, by the strict geometrical principles, is one of the few problems of elementary geometry which has never been solved. From what has been explained, it will be apparent that this problem is equivalent to that of the equisection of angles; since the subdivision of the angular space surrounding the centre of a circle necessarily infers the corresponding subdivision of the circle itself. Although the problem, in its general form, has not been solved, particular cases of it, however, admit of easy and obvious solution; thus it is evident that the circumference of a circle may be divided into four equal parts, by drawing two diameters at right angles to each other.

The four right angles thus formed, being bisected, will divide the circumference into arcs of $45°$, and these being again bisected will give arcs of $22\frac{1}{2}°$; and by continuing the process of bisection, we shall obtain arcs of the following magnitudes :—

$$\begin{array}{l} 11° \ 15'. \\ 5° \ 37' \ 30''. \\ 2° \ 48' \ 45''. \\ 1° \ 24' \ 22\frac{1}{2}''. \\ 0° \ 42' \ 11\frac{1}{4}''. \\ \&c. \ \&c. \end{array}$$

By such a process, however, it is manifest that we can never obtain an arc of the precise value of any one of the usual denominations of angular magnitude.

(100.) The most simple case of the multisection of an angle after its bisection is its trisection, or its division into three equal parts. This problem accordingly exercised, at an early epoch in the progress of geometrical science, the ingenuity of mathematicians, and has become memorable in the history of geometrical discovery, for having baffled the skill of the most illustrious geometers.

Although this celebrated problem may have lost its importance by the vast improvements made in analytical science, it may not be uninteresting to the geometrical student to be informed of the real nature of its conditions. Its object was to determine means of dividing any given angle into three equal parts by the aid of the postulates and axioms prefixed to Euclid's Elements, without any other instruments than the rule and compasses permitted by the former, and without the assumption of any other geometrical truths than those deduced by the strictest geometrical reasoning from the latter. Simple as the problem appears to be, it never has been solved, and probably never will be solved, under the above conditions.

(101.) The bisection of an angle involves other cases of the general problem of the multisection of angles. An angle being bisected, each of its parts may be again

bisected, by which it will be divided into four equal parts; and these parts being again bisected, it will be divided into eight equal parts; and by a continuation of this process of continual bisection, an angle may be divided into 16, 32, 64, &c. equal parts. In fact, it may be divided into any number of parts which can be obtained by the continual multiplication of 2. The same extent of multisection will of course be applicable to a circular arc.

(102.) In practical geometry, the problem of the multisection of an angle is attended with no difficulty. By the researches of analysis, the length of the circumference of a circle of known radius can be determined with any required degree of precision; and this being done, the length of any arc of that circle becomes a matter of easy arithmetical calculation. It is found that if the diameter of a circle were divided into a hundred equal parts, 314 such parts would be less than the circumference; and 315 of these would be greater than it. By such means the length of the circumference may be obtained to within less than one hundredth part of the diameter; but, if greater precision be required, the following table will give the means of obtaining it.

No. of Parts in the Diameter.	No. of these Parts less than Circumference.	No. of these Parts greater than Circumference.
100	314	315
1,000	3,141	3,142
10,000	31,415	31,416
100,000	314,159	314,160
1,000,000	3,141,592	3,141,593
10,000,000	31,415,926	31,415,927
100,000,000	314,159,265	314,159,266
1,000,000,000	3,141,592,653	3,141,592,654
10,000,000,000	31,415,926,535	31,415,926,536
100,000,000,000	314,159,265,358	314,159,265,359
1,000,000,000,000	3,141,592,653,589	3,141,592,653,590
10,000,000,000,000	31,415,926,535,897	31,415,926,535,898

Thus it appears that if the diameter of a circle be conceived to be divided into ten billions of equal parts,

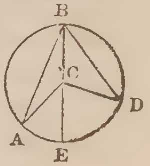
the length of its entire circumference may be determined numerically, subject to an error of less amount than one of these parts; and if this degree of accuracy were not considered sufficient, a much greater degree of precision has been attained. The diameter of the circle being taken as the unit, the number expressing the circumference of the circle has been determined to 140 decimal places. If the diameter be conceived to be divided into as many equal parts as would be expressed by 1 followed by 140 ciphers, the circumference could therefore be computed, subject to an error less in amount than one of these parts. It is needless to say that such precision greatly exceeds the exigencies of practice, and that we may consider that we are in a condition always to determine the circumference of a circle when the length of its diameter is known.

(103.) It is obvious that the same principles lead to the solution of the converse problem, to determine the diameter when the circumference is given, and the same table of numbers will suffice for this purpose. Thus, if the given circumference be conceived to consist of 314 equal parts, the diameter will be less than 100 of these parts; and if the circumference be conceived to consist of 315 equal parts, the diameter will be greater than 100 of these parts, and the same observations may be applied to the higher scales of division.

(104.) Since the whole circumference may be determined when the diameter is given, any required fractional part of it may be found; thus the 360th part of it, or the length of one degree, may be determined; and thence the fractional parts of a degree, such as minutes and seconds, may be found.

(105.) If any two points A and D, *fig.* 47., be taken in the circumference of a circle, and from those two points two straight lines be drawn to the same point B in the circumference, and other two straight lines to the centre C, the angle C, at the centre, will be twice the angle B at the circumference.

fig. 47.

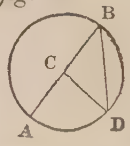


To prove this, let the line B C E be drawn, the external angle A C E of the triangle A C B will be equal to the two remote internal angles taken together (49.): but these angles are equal to each other, because the triangle B C A is isosceles (63.). Hence the angle A C E is twice the angle A B C.

In the same manner it may be proved that the angle E C D is twice the angle C B D; therefore the whole angle A C D is twice the angle A B D.

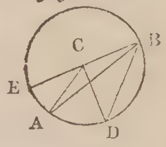
In this case, it happens that the centre C of the circle lies between the sides of the angle A B D; but it may either lie upon one of those sides or outside them. If it lie upon one of the sides, as in *fig.* 48., the angle A C D is proved to be double the angle B, in the same manner as A C E was proved to be double A B C in the last case.

fig. 48.



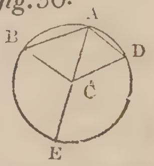
If the centre C lie outside the angle A B D, as in *fig.* 49., then the angle A C D is shown to be the difference between the angles E C A and E C D, which are respectively double the angles C B A and C B D.

fig. 49.



It may happen that the central angle is greater than 180°, as in *fig.* 50., where the arc of the circle B E D included between the sides of the angle A is greater than a semicircle. In this case, however, the proof is in all respects the same as in the first case.

fig. 50.



(106.) A straight line, joining any two points in a circle, *fig.* 51., is called the *chord of the arc* of the circle between these points; and the figure included by the chord and the arc is called a *segment of the circle.*

fig. 51.

(107.) A figure included by two radii, *fig.* 52., of a circle and the arc between them, is called a *sector* of the circle; and the angle included by the radii is called the *angle of the sector.*

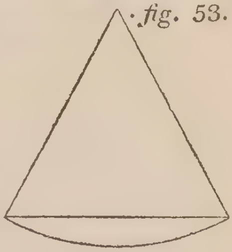
fig. 52.

(108.) It is evident, from what has been explained,

that sectors having equal radii and equal angles, must be in every respect equal, because by superposition they would cover one another.

(109.) If the ends of the radii of a sector be joined by a chord, *fig.* 53., the sector will be resolved into a segment and an isosceles triangle, the latter being formed by the radii and the chord.

fig. 53.



(110.) If lines be drawn from the ends of the chord of a segment to various points in the arc of the segment, each pair of these lines will include an angle of the same magnitude. Thus, in *fig.* 54., there are several angles formed in the segment, whose chord is A B, which angles will be all of the same magnitude; and the same would be true of any angle formed by lines drawn to any points in the same segment.

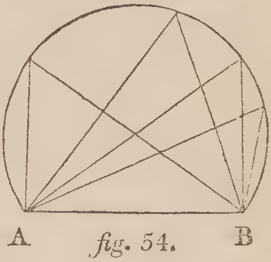


fig. 54.

This, which is one of the most remarkable and beautiful properties of the circle, follows as an immediate and obvious consequence from what has been already shown respecting the relations between corresponding angles at the centre and at the circumference.

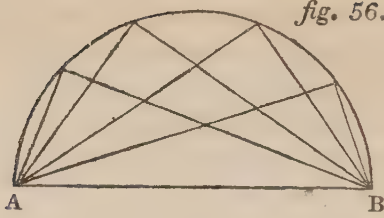
In *fig.* 55. the angles A and E are each of them half the central angle C, and consequently they are equal to each other; and the same would be true of angles formed by any other lines drawn from B and D to other points in the arc B A D.

fig. 55.

(111.) It appears, therefore, that all the angles thus formed in the same segment of a circle are equal; but it remains to be determined how this common magnitude is affected by the magnitude of the segment itself.

(112.) It is manifest that if the segment be a semicircle, *fig.* 56., the central angle, bounded by the radii, will be 180°; consequently the angle in the segment,

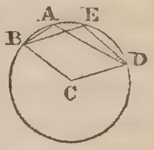
fig. 56.



being half this, must be a right angle. Hence all angles drawn in a semicircle are right angles.

(113.) If the segment be greater than a semicircle, as in *fig.* 55., the central angle will be less than 180°, therefore the angle in the segment will be acute; but if the segment be less than a semicircle, as is the case with B A D, *fig.* 57., the central angle B C D will be greater than 180°, and therefore the angle in the segment will be obtuse.

fig. 57.

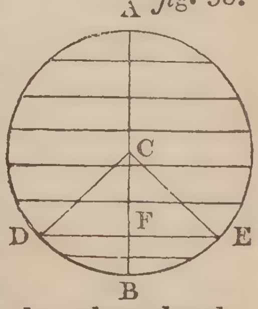


(114.) In fact, the number of degrees in the angle in a segment will be half the number of degrees in the arc of the opposite segment. Thus, in *fig.* 55., the number of degrees in the angle B A D will be half the number of degrees in the arc of the lower segment.

(115.) It has been shown, that in the same or equal circles, equal angles at the centre include equal arcs. The same will evidently be true of equal angles at the circumference, since the latter are the halves of the former.

(116.) If several parallel chords be drawn in a circle, they will be all bisected by the diameter A B, *fig.* 58. which is perpendicular to them. Let radii C D and C E be drawn to the extremities of any one of these chords, the triangle D C E, being isosceles, is divided symmetrically by the perpendicular C F (64.); consequently F is the middle point of D E: and in the same manner it may be proved that the diameter passes through the middle points of the other chords.

fig. 58.



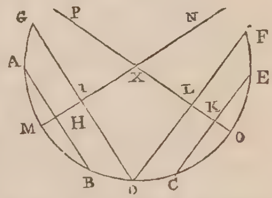
(117.) Hence it appears, that if the semicircle A D B be doubled over on the semicircle A E B, the one will precisely cover the other, since each half chord of the one semicircle will cover the corresponding half chord of the other. The diameter A B therefore divides the circle symmetrically.

(118.) Hence it appears, that if two parallel chords be drawn in a circle, the straight line passing through their middle points will be a diameter.

(119.) When the circumference of a circle is given, the centre may be found thus: — Draw two parallel chords, and through their middle points draw a straight line, terminated in the circumference: the middle point of that line will be the centre of the circle.

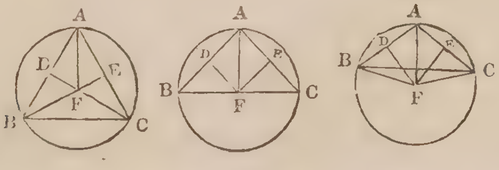
(120.) But if a part only of the circumference of a circle be given, its centre may still be found. Let it be required to find the centre of a circle a part of whose circumference is the arc G B E (*fig.* 59.); draw two parallel chords A B and G D, and finding their middle points H and I, through these points draw a straight line M N; draw other two parallel chords D F and C E, and finding their middle points K and L, through K and L draw another straight line O P: the point X, where M N crosses O P, is the centre of the circle. Since M N and O P both bisect parallel chords, each must be a diameter of the circle; and therefore the point X, where they cross each other, must be the centre.

fig. 59.



(121.) But if only three points in the circumference of a circle be given, the centre may be found, and the circle may be described.

fig. 60.



Let A, B, and C, *fig.* 60., be the three points; draw

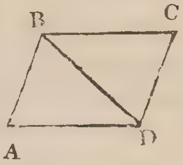
straight lines joining them, and bisecting the lines A B and A C, let their middle points be D and E; if a perpendicular to A B be drawn from D, it must pass through the centre, by what has been already proved; and in the same manner a perpendicular to A C drawn through E must likewise pass through the centre. If these two perpendiculars be drawn through D and E, the centre of the required circle will be the point F, where these perpendiculars meet. This follows from what has been already proved; but it is easy to verify it.

Since the perpendicular D F bisects the base of the triangle A F B, that triangle will be isosceles; and in the same manner the triangle A F C may be proved to be isosceles; thus B F and F C are respectively equal to A F, and the three lines therefore from F to the points A, B, and C, are equal. A circle, therefore, drawn with centre F and radius F A, will pass likewise through the points B and C.

CHAP. VI.

OF QUADRILATERAL FIGURES.

fig. 61.



(122.) In a quadrilateral figure, such as A B C D (*fig.* 61.), the angles which immediately succeed each other in going round the figure are called *adjacent angles;* and the angles which do not immediately succeed each other are called *opposite angles.*

Thus, A and B are adjacent angles; also B and C are adjacent angles. But A and C, or B and D, are opposite angles.

(123.) A line drawn in any right lined figure, joining any two angles which are not adjacent, is called a *diagonal* of the figure.

Thus in the quadrilateral (*fig.* 61.) B D is a diagonal.

(124.) A quadrilateral being resolved into two triangles by its diagonal, the sum of its four angles, being equal to the sum of the six angles of the two triangles, will be equal to four right angles.

(125.) It appears, therefore, that in four-sided as well as in three-sided figures, the aggregate amount of the angles is independent of either the length or position of the sides. In triangles this amount is always 180°; and, from what has been just proved, it follows that in quadrilaterals it is 360°.

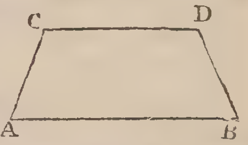
If a quadrilateral be formed of rods connected by joints or pivots at the angles, so that the shape of the figure may be varied at pleasure by changing the magnitudes of the angles, some of the angles must increase while others diminish; and the increments of those which increase must be exactly equal to the decrements of those which diminish, since, however they may vary, the gross amount of the angles must still be 360°; and the same will be true, even though the length of the rods which form the sides of the figure be altered.

(126.) If two adjacent angles of a quadrilateral figure be supplemental, the remaining angles must be also supplemental.

For, since the sum of all the angles is 360°, if two adjacent angles taken together be 180°, the remaining two must be also 180°.

(127.) If two adjacent angles of a quadrilateral be supplemental, one pair of opposite sides must be parallel.

fig. 62.

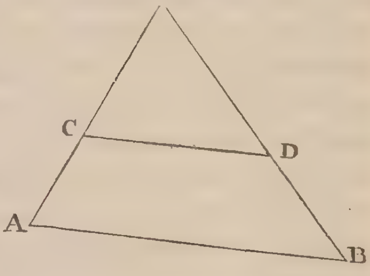


For if the angles A and C (*fig.* 62.) be supplemental, the lines A B and C D must be parallel.

(128.) Such a quadrilateral is called a *trapezium;* the parallel sides are called its bases; and the sides not parallel, A C and B D, are called its sides.

(129.) A trapezium may be considered as produced by cutting off the upper part of a triangle by a line parallel to its base.

fig. 63.



Thus in *fig.* 63., if the line C D be drawn parallel to the base A B, A C D B will be a trapezium.

(130.) When a portion is cut from the upper part of a figure in this manner, the figure is said to be *truncated.*

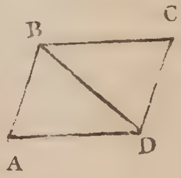
Thus a trapezium is a *truncated triangle.*

(131.) If the angles adjacent to one base of a trapezium be equal, those adjacent to the other base must also be equal.

For if A and B (*fig.* 62.) be equal, their supplements (126.) C and D must also be equal.

(132.) A quadrilateral figure in which both pairs of opposite sides are parallel, is called a *parallelogram.*

fig. 64.



Thus in *fig.* 64., if A B be parallel to D C, and A D parallel to B C, the figure will be a parallelogram.

(133.) In a parallelogram the ad-

jacent angles are supplemental, and the opposite angles are equal.

Since A D (*fig*. 64.) is parallel to B C, A is the supplement of B, and D is the supplement of C; and since A B is parallel to D C, A is the supplement of D, and B is the supplement of C.

The angles A and C are equal, because each of them is supplemental to the angle B; and the angles B and D are equal, because each of them is supplemental to the angle C.

(134.) The triangles into which a parallelogram is resolved by either of its diagonals, are in all respects equal.

For the angle A B D (*fig*. 64.) is equal to the angle C D B, since they are alternate angles (42.); and, for the same reason, the angle A D B is equal to the angle C B D. In the two triangles, therefore, the side B D is common, and the angles between which it lies are respectively equal; therefore, the side A B is equal to the side C D, and A D to C B, and the triangles are in all respects equal (61.)

(135.) In a parallelogram the opposite sides are equal.

This has been proved in the last case.

(136.) If each pair of opposite angles of a quadrilateral be equal, the figure must be a parallelogram.

For if the angles A and B (*fig*. 64.) be respectively equal to the angles C and D, they will be half the sum of the angles of the figure, and will therefore be equal to 180° (125.); and, therefore, the sides A D and B C will be parallel (40.). In the same manner it may be shown that the angles B and C are together equal to 180°; and therefore the sides A B and D C are parallel.

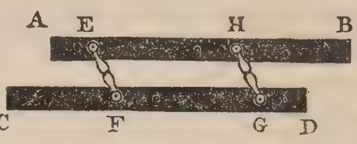
(137.) If each pair of opposite sides of a quadrilateral be equal, the quadrilateral will be a parallelogram.

For if A B be equal to C D (*fig*. 64.), and B C to A D, the two triangles into which the figure is resolved

by the diagonal, will have the three sides in the one respectively equal to the three sides in the other, and therefore their angles will be equal each to each: since the angle C B D is equal to the angle A D B, the side A D is parallel to B C; and since the angle A B D is equal to the angle C D B, the side A B is parallel to the side C D.

(138.) Upon this principle are constructed instruments used in geometrical and mechanical drawing, called *parallel rulers.*

fig. 65.



In *fig.* 65. A B and C D are two rulers; E F and G H are two pieces of brass equal in length, fastened on pins at equal distances, G F and H E, on each of the rulers, and capable of turning on those pins. The two rulers may be moved to different distances from each other, but will always be parallel. Thus, if the edge of one ruler be placed along a straight line, a pen drawn along the edge of the other will trace a parallel straight line. The accuracy of this instrument depends on the circumstance of the distance between the pins on each of the rulers being exactly equal, and on the exact equality of the bars E F and G H.

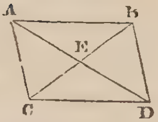
(139.) Although the triangles into which a parallelogram is resolved by its diagonal be equal in all respects, yet the diagonal does not divide the figure symmetrically, because the *position* of the triangles on either side of the diagonal is reversed. If the triangle B A D (*fig.* 64.) be folded over along the diagonal upon the triangle B C D, the point A would not fall upon the point C.

(140.) The diagonals of a parallelogram bisect each other.

For since the sides A C and B D (*fig.* 66.) are equal, and also the angles C A E and B D E as well as A C E and D B E, the sides (61.) C E and B E and also A E and E D are equal.

(141.) If the diagonals of a quadrilateral bisect each other, it will be a parallelogram.

fig. 66.



For since A E and E C (*fig.* 66.) are respectively equal to D E and E B, and the angles A E C and D E B are also equal (20.), the angles A C E and D B E are equal (59.); and therefore the lines A C and B D are parallel (43.); and in like manner it may be proved that A B and C D are parallel.

(142.) If one angle of a parallelogram be right, all the angles must be right.

For if one angle be right, the angle opposite must also be right, since they must be equal (133.); and the angles adjacent must be right, since they must be supplemental to the former (133.).

fig. 67.

(143.) A right-angled parallelogram is called a *rectangle*. (*fig.* 67.)

(144.) The diagonals of a rectangle are equal.

For the adjacent angles A and B (*fig.* 68.) are equal, being right, and the opposite sides A C and B D are equal (135.); and the side A B is common to the two triangles C A B and A B D, and therefore (59.) the diagonals A D and C B are equal.

fig. 68.

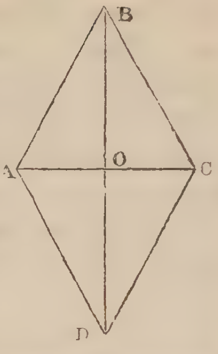
(145.) A parallelogram of which all the sides are equal (*fig.* 69.) is called a *lozenge*.

(146.) The diagonals of a lozenge bisect its angles.

For since A B C is an isosceles triangle; the angles B A C and B C A are equal (63.); and since B C is parallel to A D, the angle B C A is equal to the alternate angle D A C (42.); therefore the angle B A C is equal to the angle D A C; and therefore the diagonal A C bisects the angle B A D.

In the same manner it may be proved that A C bisects the angle B C D, and that B D bisects the angles A B C and A D C.

fig. 69.



(147.) The diagonals of a lozenge intersect each other at right angles.

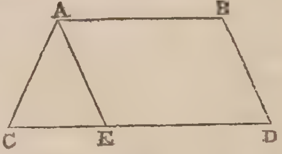
For since B D (*fig.* 69.) bisects the angles A B C and A D C (146.), if the triangle B A D be doubled over along the line B D upon the triangle B C D, the side B A will fall upon B C, and D A upon D C; therefore the point A will fall upon the point C, and the line O A upon the line O C. The angle B O A, therefore, exactly covers the angle B O C, and is therefore equal to it; and the angle A O D covers the angle C O D, and is therefore equal to it; the angles at O are therefore right angles.

(148.) Each of the diagonals of a lozenge divide the figure symmetrically.

For it appears by what has been already proved (147.) that the triangles into which the lozenge is divided are precisely equal, and admit of superposition.

(149.) If the sides of a trapezium be equal, they will form equal angles with its bases.

fig. 70.



For let A E (*fig.* 70.) be drawn, parallel to B D; then the figure A B D E will be a parallelogram, and therefore A E will be equal to B D. But B D is equal to A C; therefore A E is equal to A C. The triangle C A E is therefore isosceles, and the angle A E C is equal to the angle C; but since A E is parallel to B D, the angle A E C is equal to the angle D; therefore the angle C must be equal to the angle D, which are the angles that the sides make with the base C D. Again since A B is parallel to C D, the angles B A C and C, as well as B and D, are supplemental (41.); but the angle C has been proved to be equal to the angle D; therefore the angle C A B is equal to the angle B, which are the angles that the sides make with the base A B.

(150.) If the angles at the base of a trapezium be equal, its sides will be equal.

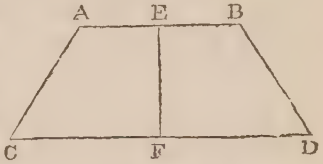
For, let A E (*fig.* 70.) be drawn parallel to B D; then, as in (149.), A E will be equal to B D; but since the angle A E C is equal to the angle D, it is also equal to the angle C, and therefore the triangle C A E is isosceles, and the side C A is equal to the side A E; but A E has been proved equal to B D, and therefore A C is equal to B D.

(151.) A trapezium of this kind is called a *symmetrical trapezium.*

(152.) The line joining the middle points of the bases of a symmetrical trapezium divides the figure symmetrically.

Let E (*fig.* 71.) be the middle point of the base A B, and let E F be drawn perpendicular to A B; if the figure be now folded along the line E F, so that that part to the right of E F shall be turned over upon that part to the left, the line E B will fall upon the line E A, because of the equality of the right angles at E. The point B will fall upon the point A, because E B is equal to E A. The side B D will fall upon the side A C, because of the equality of the angles B and A (151.) The point D will fall upon the point C, because B D is equal to A C; and since the point D falls upon the point C, the line F D must coincide with the line F C, and must therefore be equal to it. The angles at F will also lie one upon the other, and are therefore equal, and being equal are right angles.

fig. 71.



It is evident, therefore, that the figure is symmetrically divided by the line E F.

(153.) A rectangle is divided symmetrically by lines joining the middle points of its opposite sides.

This may be proved in the same manner as the corresponding property of the symmetrical trapezium was established (152.)

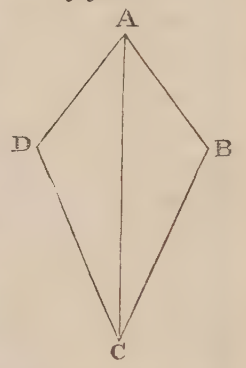
(154.) A square is symmetrically divided by each of

its diagonals, and also by lines joining the middle points of its opposite sides.

For a square being a lozenge, and also a rectangle, what was proved in (147.) and in (153.) are applicable to it.

(155.) If a quadrilateral have the sides containing two opposite angles equal, the diagonal drawn between those angles will divide the figure symmetrically.

fig. 72.



For since A D is equal to A B (*fig.* 72.), and C D to C B, the triangles into which the quadrilateral is resolved, having their sides respectively equal, will have their corresponding angles also equal; therefore the angles at A and C will be each bisected by the diagonal. If the triangle A B C be folded along the diagonal over A D C, the side A B will fall upon the side A D, because the angle C A B is equal to the angle C A D; and the side C B will fall upon the side C D, because the angle A C B is equal to the angle A C D; therefore the triangles will exactly cover one another, and therefore the figure is divided symmetrically by the diagonal A C.

CHAP. VII.

OF INSCRIPTION AND CIRCUMSCRIPTION OF FIGURES.

(156.) A FIGURE which has the vertices of its several angles in the circumference of the same circle is said to be *inscribed* in that circle; and the circle is said to be *circumscribed about* such a figure.

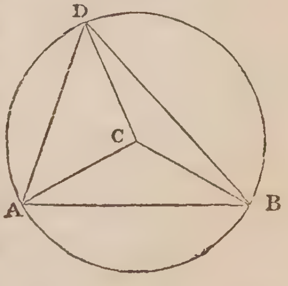
(157.) A figure each of whose sides is a tangent to the same circle is said to be *circumscribed about* that circle; and the circle is said to be *inscribed* in such a figure.

(158.) A circle may be circumscribed round any given triangle; for, it has been shown (121.) that a circle may always be described passing through three given points, provided these three points do not lie on the same straight line.

(159.) A triangle having its three angles given, may be inscribed in a circle.

As the three angles of the triangle must, taken together, be equal to 180°, angles of twice their magnitude must be equal, taken together, to 360°. Let the space round the centre C (*fig.* 73.) of the proposed circle be divided by three radii, C A, C D, and C B, into three angles, which shall respectively be double the three angles of the triangle, and let lines be drawn joining the points A, B, and D. These lines will form the required triangle. For the angle at A is half the angle B C D (105.), and is therefore equal to one of the angles of the proposed triangle; and, in the same manner, the angles at B and D are respectively halves of the angles A C D and A C B, and are therefore the other angles of the required triangle.

fig. 73.

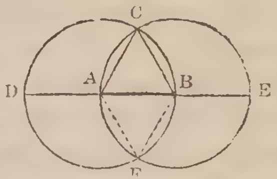


(160.) It is evident that this problem is equivalent to the division of the circumference of a circle into three parts of given magnitudes, inasmuch as the three arcs of the circle, of which the three sides of the triangle are chords, consist of twice as many degrees respectively as are contained in the angles of the triangle.

(161.) To inscribe an equilateral triangle in a circle, it is only necessary to draw three radii from the centre, making with each other angles of 120°, the angles of an equilateral triangle being 60° (72.).

(162.) To construct an equilateral triangle, whose side shall be of a given length.

fig. 74.

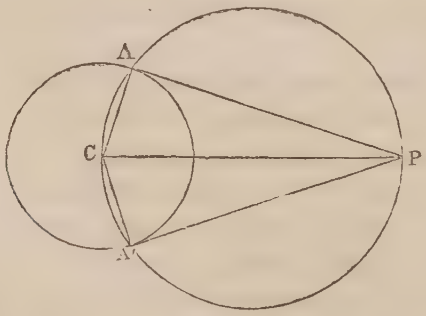


Let the line A B (*fig.* 74.) be the length of the side of the proposed triangle; and with B as centre and B A as radius, and with A as centre and A B as radius, let two circles be described; lines drawn from A and B to either of the points C or F, where these circles intersect, will form with the line A B an equilateral triangle.

It is evident that the triangles thus constructed are equilateral, since their sides are the radii of equal circles.

(163.) To draw a tangent to a circle from a point outside it.

fig. 75.



Let P (*fig.* 75.) be the point from which the tangent is to be drawn; draw a line from P to the centre C of the given circle, and on P C as a diameter describe

a semicircle. To the point A, where this semicircle crosses the given circle, draw P A. This line P A will be the required tangent. For if C A be drawn, the angle C A P, being an angle in a semicircle, will be a right angle (112.); and therefore P A must be a tangent (83.).

Since a semicircle may be described either above or below the line P C, it follows that two tangents may be drawn to the circle from the point P.

(164.) The angle A C A′ included between the two radii drawn from C (*fig.* 75.) to the points of contact of the tangents, will be the supplement of the angle A P A′ included by the tangents themselves.

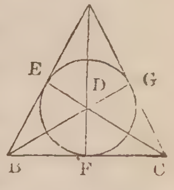
For in the quadrilateral C A P A′, formed by the tangents and the radii, the four angles taken together are equal to four right angles (125.); but the angles at A and A′ being right angles, the angles at P and C, taken together, must be equal to two right angles, and must therefore be supplemental.

(165.) When two tangents are drawn from the same point P (*fig.* 75.) to the same circle they will be equal, and the line drawn to the centre will bisect the angle formed by them.

For if the triangle C A′ P below the line P C be folded over upon the triangle C A P above it, the right angle of the one must fall upon the right angle of the other, and the triangles must coincide in every respect; and therefore the sides and angles must be respectively equal.

(166.) To inscribe a circle in a triangle.

fig. 76.



Let A B C be the proposed triangle; since B A and B C must be tangents to the inscribed circle (157.), the line B D bisecting the angle A B C must pass through the centre of the circle (163.); and, for the same reason, the line C D bisecting the angle A C B must pass through the same centre. Hence, if these lines be drawn bisecting the two angles A B C and A C B, the point D where

these two bisectors meet will be the centre of the inscribed circle; if perpendiculars be drawn from D to the three sides, these perpendiculars will be radii of the inscribed circle.

(167.) To circumscribe about a given circle a triangle whose angles shall have given magnitudes.

From the centre D (*fig.* 76.) let three radii, D E, D F, and D G, be drawn, dividing the space round the centre into angles which shall be respectively the supplements of the angles of the required triangle. Through the extremities G, E and F, of these radii let tangents be drawn; these tangents will form the required triangle. For the angles included by the tangents respectively being the supplements of the angles contained by the corresponding radii (164.), will be the angles of the required triangle.

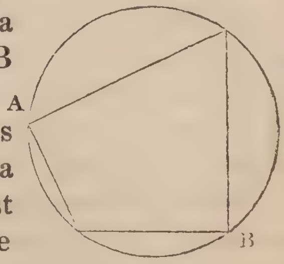
(168.) To circumscribe an equilateral triangle about a circle, it will only be necessary to draw three radii at angles of 120° with each other. Tangents through their extremities will form a triangle whose angles will be 60° (164.), which will therefore be equilateral (72.).

(169.) If a quadrilateral figure be inscribed in a circle, its opposite angles will be supplemental.

For each such angle will consist of half the number of degrees contained in the opposite arc of the circle (114). Therefore the two opposite angles, taken together, must be equal to half the number of degrees contained in the whole circumference, and must therefore be equal to 180°.

(170.) Hence if one angle, A (*fig.* 78.), of a quadrilateral inscribed in a circle be right, the opposite angle B must also be right.

fig. 78.



(171.) If two adjacent angles of a quadrilateral inscribed in a circle be right, all the angles must be right, since the others must be their supplements.

(172.) No parallelogram can be inscribed in a circle except a rectangle.

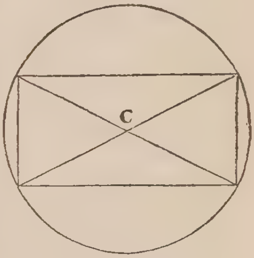
For the opposite angles of every quadrilateral inscribed in a circle must be supplemental (169.), and the opposite angles of a parallelogram must be equal (133.). To be at the same time equal and supplemental, the angles must therefore be right, and the figure must be a rectangle.

(173.) The diagonals of any rectangle inscribed in a circle must be diameters of the circle.

For the angles contained in the segments of which these diagonals are chords being right angles, the segments must be semicircles (112.).

(174.) If any two diameters in a circle be drawn, the lines joining their extremities will form an inscribed rectangle (*fig.* 79.)

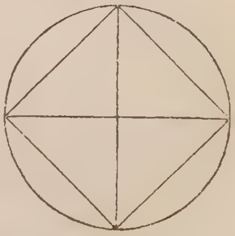
fig. 79.



(175.) If two diameters be drawn at right angles, the lines joining their extremities will form an inscribed square.

By what has been already proved, it will be evident that the figure will be a rectangle; and, since the central angles are right angles, the arcs of the circle are equal, and therefore their chords are equal, and the figure is therefore a square. (*fig.* 80.)

fig. 80.



(176.) If tangents to a circle be drawn through the ends of the same diameter, they will be parallel.

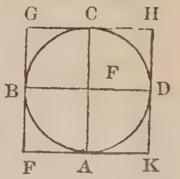
For they will be both perpendicular to the diameters.

(177.) If two diameters of a circle be drawn at right angles, tangents through their extremities will form a circumscribed square.

For the figure will be a parallelogram, since its opposite sides will be parallel (132.); and it will be rectangular, since its sides are parallel to the rectangular dia-

meters C A and B D (*fig.* 81.); finally, its sides will be equal, since they are opposite sides in the the same parallelograms with the diameters of the circle.

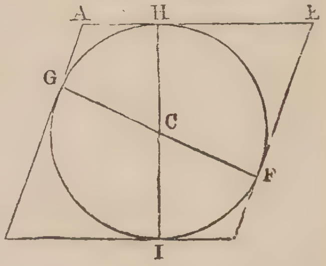
fig. 81.



(178.) The sides of the circumscribed square are therefore equal to the diameters of the circle.

(179.) If two diameters of a circle be drawn at any proposed angle, tangents through their extremities will form a circumscribed parallelogram, whose angles shall be equal to the angles contained by the diameters.

fig. 82.



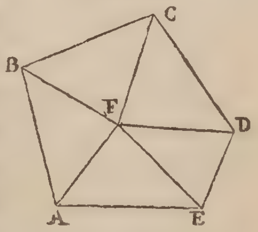
For let G F and H I (*fig.* 82.) be the two diameters. The angle A is the supplement of the angle G C H, and therefore equal to the angle H C F; and the angle E is the supplement of the angle H C F (164.), and therefore equal to the angle H C G.

(180.) Hence a parallelogram having any required angles may always be circumscribed round a circle by drawing two diameters making with each other the angles of the proposed parallelogram, and through their extremities drawing tangents.

(181.) Right-lined figures consisting of more than four sides are usually called *polygons*.

A right-lined figure having all its sides and angles equal is called a *regular polygon*.

fig. 83.



(182.) If a point F (*fig.* 83.) be taken within a polygon, and lines be drawn from it to the several angles A, B, C, D, E, the figure will be resolved into as many triangles as there are sides. The angles of these triangles, taken together, will be equal to twice as many right angles as there are sides. But, with the exception of the angles round the point F, these angles compose the angles of the polygon.

The angles round the point F are together equal to four right angles; hence it follows, that if to the sum of all the angles of any polygon, four right angles be added, we shall obtain twice as many right angles as the figure has sides.

(183.) Hence the sum of all the angles of any right-lined figure, whose number of sides is given, will be found by taking twice as many right angles as the figure has sides, and deducting four.

This general proposition includes the two which have been already proved, respecting the angles of triangles and quadrilaterals (47.) (124.). If the number of sides of a triangle be doubled, and four be substracted, the remainder will be two; and if the number of sides of a quadrilateral be doubled, and four be substracted, the remainder will be four. Thus it would follow, that the angles of a triangle will be equal to two right angles, and those of a quadrilateral to four.

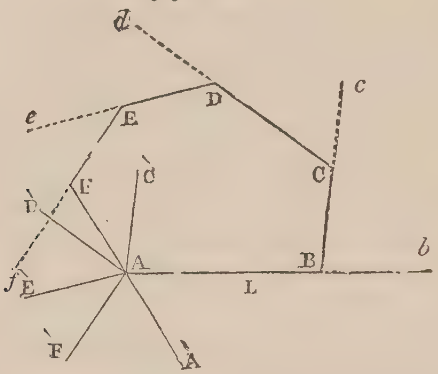
(184.) In general, if the number of sides of the figure be expressed by the number in the first line of the following table, the number of right angles, to which the sum of its angles will be equal, will be expressed by the number in the second; and the sum of its angles in degrees, in the third.

Number of sides	3	4	5	6	7	8	9	10	11	12	13	14
Number of right angles	2	4	6	8	10	12	14	16	18	20	22	24
Sum of angles	180°	360°	540°	720°	900°	1080°	1260°	1440°	1620°	1800°	1980°	2160°

(185.) The remarkable property which has been already noticed in figures with three and four sides, in virtue of which the sum of their angles continues the same, however they may change as to the length and position of their sides, is therefore a general property of all right-lined figures. So long as the number of sides remains unaltered, so long will the sum of the angles remain the

same, however the sides or angles individually may be varied in their magnitudes. It may not be uninteresting to trace this very remarkable property more immediately to its origin, than is done by the method of investigation which we have pursued in its demonstration.

fig. 84.



Let us take, for example, any figure, such as A B C D E F (*fig.* 84.); and suppose, at the angle A, a line A L placed, capable of revolving on A as a pivot or centre; if A L then be supposed to turn round the point A until it take a position A C′ parallel to B C, it will revolve through an angle B A C′, equal to the external angle *b* B C. If it be again supposed to turn from the position A C′, until it take the position A D′ parallel to C D, it will revolve through the angle C′ A D′ equal to *c* C D. Again, if it turn from the position A D′ till it take the position A E′ parallel to D E, it will revolve through another angle D′ A E′ equal to *d* D E. If it again revolve till it take the position A F′ parallel to E F, it will turn through the angle E′ A F′ equal to the external angle *e* E F. If it move from the position A F′ till it take the direction A A′ of the side F A, it will have moved through an angle F′ A A′ equal to the external angle *f* F A; and finally if it revolve from the position A A′ till it coincide with A B it will turn through the angle A′ A B; and it will thus have made one complete revolution round the point A, moving, as it

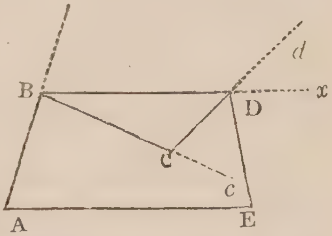
revolves successively, through angles equal to the several external angles of the figure. It is obvious, therefore, since all the angles round the point A, taken together, are equal to four right angles, that all the external angles of the polygon must also be equal to four right angles.

(186.) It is evident that each angle of the figure being the supplement of its adjacent external angle, the internal and external angles, taken together, will be equal to twice as many right angles as the figure has sides; but, from what has been already shown, the external angles alone are equal to four right angles.

(187.) Thus, the number four, which is deducted from double the number of sides, in computing the aggregate value of the angles of the figure, may be considered as representing the gross amount of the external angles.

(188.) To this reasoning there is, however, an exception. In *fig.* 84. the case contemplated is the case of what is called a *convex figure*. To make the import of this term intelligible, it must be remembered that two lines may be considered as forming an angle greater than two right angles; and such may be the internal angle of any right-lined figure which has more than three sides.

fig. 85.



Thus in *fig.* 85. the angle B C D within the figure is greater than 180° by the magnitude of the angle *c* C D. In this case the internal angles of the figure are computed in the same manner as has been already explained, and the demonstration given in (182.) will still be applicable. But the sum of the external angles will be greater than four right angles by the magnitude of the angle *c* C D. This will be evident if we draw the line B D. The figure A B D E, having no angle greater than 180° will have the sum of its external angles equal to four right angles. But in the figure A B C D E, the external angles are greater than those of A B D E, by

the two angles x D d or B D C and D B C, taken together. But these two latter angles C D B and D B C are together equal to the angle c C D. The exterior angles, therefore, of the figure A B C D E exceed four right angles by the magnitude of the angle by which the convex angle B C D exceeds two right angles.

(189.) If a right-lined figure have one or more convex angles, these angles have no adjacent external angle, and each of them exceeds two right angles by a certain excess, while each concave angle, with its adjacent external angle, is equal to two right angles. From this way of considering figures which have convex angles we may also deduce the amount of the sum of the external angles; for, the sum of all the angles internal and external including the convex angles, is equal to twice as many right angles as the figure has sides, together with the excess of every convex angle above two right angles. But the sum of the internal angles alone falls short of twice as many right angles as the figure has sides by four. Hence the sum of the external angles must be equal to those four right angles, together with the excess of every convex angle above two right angles.

CHAP. VIII.

OF REGULAR POLYGONS.

(190.) Among the innumerable varieties of form which polygons present to the contemplation of the geometer, those which deserve most attention, as well on account of their connection with other parts of geometry, as on account of their intrinsic beauty and their application in the arts, are the regular and symmetrical polygons.

(191.) A regular or symmetrical right-lined figure is one of which all the angles and all the sides are equal.

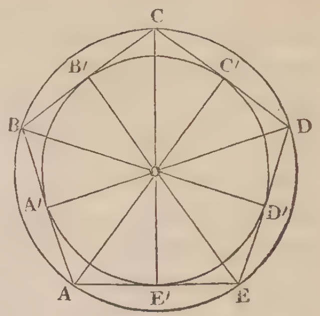
The equilateral triangle and the square, are the symmetrical figures of three and four sides.

(192.) If straight lines be drawn bisecting the several angles of a regular polygon, these lines will meet at a common point within the polygon; and that point will be equally distant from all its angles; and will therefore be the centre of a circle, which may be circumscribed around it. And it will also be equally distant from the several sides; and will therefore be the centre of a circle which may be inscribed in the polygon.

To prove this, let A B C D E (*fig.* 86.) be a regular polygon; and let lines be drawn bisecting the angles A and B, and let these lines meet at O. If from the point O, a straight line be drawn to the vertex of the angle C, that line will bisect the angle C; and the lines O C, O B, and O A, will be equal to each other. For, since the angles A and B are equal (191.), their halves are equal; therefore, the angle O A B is equal to the angle O B A, and therefore the side O A is equal to the side O B. Also, since the angle O B C is equal to the angle O B A, the

side B C equal to the side B A, and the side B O common, the triangles C B O and A B O are in all respects equal (59.); therefore O C is equal to O A, and therefore, also, to O B. But the angle O A B is half the angle E A B, therefore the angle O C B is also half the angle E A B; but the angle E A B is equal to the angle B C D; therefore the angle O C B is half the angle B C D, and therefore O C bisects the angle B C D. In the same manner it may be proved that O D is equal to O A and O B, and that it bisects the angle C D E; and, in like manner, every line drawn from O to the vertex of any angle of the polygon, may be proved to be equal to O A or O B, and to bisect the angle of the polygon.

fig. 86.



Since the lines from O to the vertices of the angles severally are equal, a circle described with the point O as centre, and any one of these lines as radius, must pass through the vertices of all the angles of the polygon, and will be a circumscribed circle.

If from the same point O perpendiculars be drawn to the several sides of the polygon, these perpendiculars will be equal.

Let O A′ and O B′ be two such perpendiculars, drawn from O to the sides A B and B C, these perpendiculars will be equal; because the triangles A O B and B O C having been already proved to be equal, so as to admit of superposition, the perpendiculars O A′ and O B′ will bisect the sides A B and B C; and therefore if B A be conceived to be turned over on B C, the point A′ will fall upon the point B′, and the perpendicular A′ O will fall upon the perpendicular B′ O, and will therefore be equal to it; and, in the same manner, it may be proved that all the other perpendiculars from the point O, upon the sides, severally are equal.

If the point O be taken as a centre, a circle described

with any one of the perpendiculars as radius will pass through the point where the perpendiculars severally meet the sides; and, since the sides are perpendicular to the radii of the circle, they will be tangents to it: the circle is therefore inscribed in the polygon, and the polygon is circumscribed around the circle.

(193.) It has been proved that the magnitude of all the angles of a polygon, taken together, is found by multiplying two right angles, or 180°, by a number which is two less than the number of sides of the figure (183.); or if n be the number of sides, then the gross magnitude of the angles, taken together, would be found by multiplying 180° by $n-2$.

But since the angles of a regular polygon are equal to each other, the magnitude of each of them will be found by dividing the total magnitude of the angles by the number, or, what is the same, by the number of sides. Hence, if n, as before, be the number of sides, the magnitude of each angle will be found by multiplying 180° by $n-2$, and dividing the product by n; or, what is the same, divide 360° by the number of sides, and subtract the quotient from 180°, the remainder will then be the magnitude of the angles.

Hence the magnitude of the angles of the regular figures, from the equilateral triangle upwards, may be computed as in the following table:—

Number of sides	3	4	5	6	7	8	9	10	11	12
Magnitude of angle	60°	90°	108°	120°	$128\frac{4}{7}$°	135°	140°	144°	$147\frac{3}{11}$°	150°

(194.) It is evident that no regular polygons can have angles consisting of a whole number of degrees, except when the number of sides is an exact divisor of 360. It appears, therefore, from what has been already shown respecting the divisors of 360 (13), that there are only twenty-one regular figures, whose angles are expressed by a whole number of degrees.

(195.) In ornamental architecture polygons are used in the formation of surfaces produced by the juxtaposition of solid blocks, as in flooring, paving, or by their superposition as in masonry. The polygons, used in such cases, must always be such as will admit of being put together without leaving open spaces between them. If they be laid together, as is sometimes the case, leaving the vertices of their angles coincident, then no regular figures can be used, except those whose angles are of such a magnitude, as will exactly fill the space surrounding a point. It is evident that the equilateral triangle and the square will fulfil this condition; since six angles of an equilateral triangle, and four of a square, make up exactly 360°; thus the point O in *fig* 87. is surrounded by six equilateral triangles, and in *fig.* 88. it is surrounded by four squares.

fig. 87.

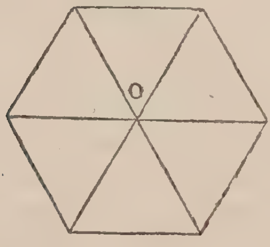
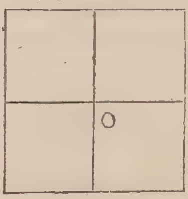


fig. 88.

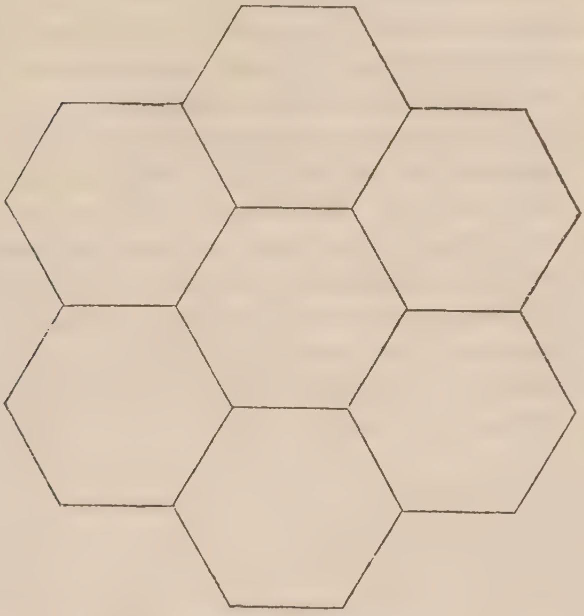


In general, the condition necessary to be fulfilled is, that $180° - \frac{360}{n}°$ should divide 360° without a remainder; or, if we divide both of these by 180°, the condition will be that $1 - \frac{2}{n}$ shall divide 2 without a remainder. The only whole numbers which will fulfil this condition are 3, 4, and 6; and it follows that a surface cannot be completely covered by any regular figures except by the equilateral triangle, the square, and the hexagon.

The angles of the hexagon being 120°, three of them will fill the space round a point. This arrangement is represented in *fig.* 89.

In the formation of pavement, it is an object to avoid the combination of a great number of angles at

fig. 89.



the same point; the strength of the surface being thereby weakened, and the liability to fracture increased. The combination of equilateral triangles, represented in *fig.* 87. is objectionable on these grounds; and, even the combination of squares, represented in *fig.* 88., is usually avoided in square pavements, by causing the angles at which each pair of adjacent sides are united, to coincide with the middle of the sides of a succeeding series, as represented in *fig.* 90. Where the angles of the com-

fig. 90.

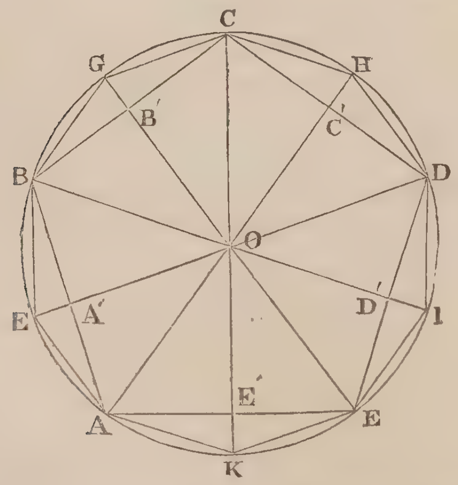
ponent figures are intended to be invariably combined, the hexagonal arrangement will therefore have greater strength and stability for pavement than the others; but for upright masonry the square or rectangular division is preferable, since the surfaces of contact take the position best adapted to sustain the incumbent weight of the structure.

(196.) Six equilateral triangles placed so as to surround the same point, *fig.* 87., will evidently form a regular hexagon; for the sides of the figure being the six bases of the equilateral triangles opposite the point O, at which they are united, are equal; and the angles of the figure being each twice the angle of an equilateral triangle, are likewise equal. Hence the figure is a regular hexagon; and, in this way, the construction of the regular hexagon depends on that of the equilateral triangle.

Any regular figure having been constructed and circumscribed by a circle, another regular figure with twice the number of sides, may be drawn.

For let A B C D E *fig.* 91. be the former figure cir-

fig. 91.



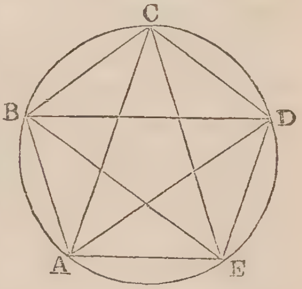
cumscribed by a circle; and let perpendiculars from the centre O be drawn to the several sides and produced

to meet the circle; these perpendiculars will bisect the angles, formed by lines drawn from the centre O to the angles of the polygon, as is obvious from what has been already proved in (192.). They will therefore bisect the arcs of the circle, of which the sides of the polygon are chords; and the circle will, therefore, be divided into twice as many equal arcs as before; the chords of which, being drawn, will be equal, and will include equal angles; and will, therefore, form a regular polygon with twice as many sides as those of the first polygon.

(197.) Hence the construction of the square leads to that of the octagon; the construction of the pentagon leads to that of the decagon; and so on.

(198.) Each diagonal of a regular pentagon cuts off an isosceles triangle, the vertical angle of which is triple its base angle: for, let A B C D E (*fig.* 92.) be a regular pentagon circumscribed by a circle; and let the diagonals B D, A C, and C E, be drawn; the angles B C A, A C E, and E C D, are equal, since they stand on equal arcs (115.): therefore the angle B C D is three times the angle B C A. But the angles B C A and C B D stand on equal arcs, and are therefore equal; therefore the angle B C D is three times the angle C B D.

fig. 92.



(199.) The other diagonals of the figure being drawn, it is evident that the angles B D A and C B D are equal, since they stand on equal arcs (115.); therefore the diagonal A D is parallel to the side B C (40.); and, in like manner, it may be proved that each diagonal of the pentagon is parallel to the side not contiguous to it; thus B D is parallel to A E, C E to B A, A C to D E, and B E to C D.

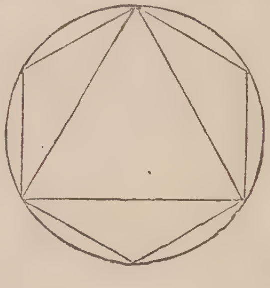
(200.) In a regular pentagon each pair of diagonals which do not cross each other, form an isosceles triangle, whose base angle is twice its vertical angle.

For the angles E A D and D A C are equal, since they stand on equal arcs (115.); and therefore the angle E A C is twice the angle E A D. But the angles E A D and A C E are equal, since they stand on equal arcs; therefore the angle E A C, at the base of the isosceles triangle A C E, is double the angle A C E at its vertex.

(201.) The side of a regular hexagon is equal to the radius of the circle circumscribing it.

For it has been already proved (196.) that the hexagon is composed of six equilateral triangles, having a common vertex at the centre of the figure, and having the sides of the figure for their six bases (*fig.* 87.). The sides of these triangles are radii of the circum-scribing circle, and are equal to the sides of the hexagon, which are bases of the same equilateral triangles.

fig. 93.



(202.) The three diagonals of a regular hexagon which do not intersect each other, form an equilateral triangle inscribed in the same circle with the hexagon (*fig.* 93).

CHAP. IX.

OF THE AREAS OF FIGURES.

(203.) The magnitude or extent of space included within the linear boundaries of any figure is called its *area.*

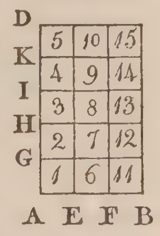
(204.) It is usual to express the area of a figure numerically by resolving it into equal squares, the sides of the squares being the linear unit; thus, if the linear unit be an inch, the area of the figure will be expressed by stating the number of square inches of which it consists; if the linear unit be a foot, the area will be expressed in square feet; and so on.

(205.) It is apparent that the square of the linear unit is the superficial unit.

(206.) If the sides of a rectangle be divided into linear units, the number of superficial units in its area will be found by multiplying the number of linear units in its base by the number of linear units in its height.

From the points where the base is divided into linear units, let parallels be drawn to the height. These parallels will resolve the area into as many oblong rectangles as there are units in the base; and these oblong rectangles will be equal to each other, since their sides are equal. Let parallels to the base A B (*fig.* 94.) be now drawn from the points where the height A D is divided into linear units. These parallels will divide each of the former oblong rectangles into as many squares of the linear unit as there are linear units in the height A D. For, each of the oblong rectangles corresponding to the units of the base, there are therefore as many squares as there are

fig. 94.



units in the height; therefore the number of squares will be found by multiplying the number of units in the base by the number of units in the height.

Thus the base AB consists of three units; and the parallels to A D, from E and F, resolve the figure into three oblong rectangles standing on the bases A E, E F, and F B. Each of these rectangles is resolved into five squares by the parallels to A B from the points of division of A D. Thus the number of units in A B being three, and in A D five, the number of squares composing the area of the rectangle, is 15.

In general, therefore, when the sides of a rectangle are given in numbers, its area is expressed by the product of the numbers representing its sides.

Thus, if a rectangular room be 20 feet long and 15 feet wide, its floor will consist of 20×15, or 300 square feet.

(207.) Hence, if the area of a rectangle be given in numbers, and one side of it be known, the other side may be found by dividing the area by the known side.

Thus, if it be given that the area of a rectangle is 300 square feet, and that one side of it be 20 feet, the other side must evidently be that number which, multiplied by 20, would produce 300; and that number is found by dividing 300 by 20, and is 15.

(208.) If the base and height of an oblique parallelogram be equal respectively to the sides of a rectangle, the area of the parallelogram will be equal to that of the rectangle.

Let E F G H (*fig.* 96.) be the oblique parallelogram, and A B C D (*fig.* 95.) be the rectangle; the base E F

fig. 95.

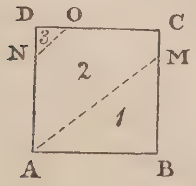
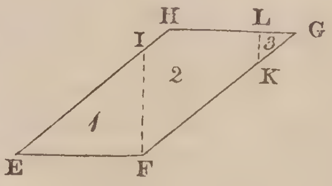


fig. 96.



being equal to A B, and the height of the parallelogram equal to A D, then the area of E F G H will be equal to A B C D.

For, from F draw F I perpendicular to E F, and take B M equal to F I, and draw A M. Since E F and F I are equal respectively to A B and B M, and the included angles are right, the triangle E F I will exactly cover the triangle A B M. Take F K equal to E I, and therefore to A M; and draw K L parallel to F I, and therefore perpendicular to E F, and therefore also to H G; it is evident that a line joining I and K would be parallel to E F, since F K and E I are equal and parallel. Hence the height of the parallelogram E F G H is equal to F I and K L taken together. Take A N equal to F I, and draw N O parallel to A M: since A D is equal to the height of the parallelogram E F G H, and F I is equal to A N or B M, L K must be equal to D N or C M. The angle K L G being right is equal to the angle D; and the angle G will be equal to the angle E (133.), and therefore to the angle M A B, and therefore to the angle D O N, since N O is parallel to A M. In the triangles K L G and N D O the angles L and G are equal respectively to D and O, and the side L K is equal to the side D N; therefore the triangles, being placed one upon the other, would exactly cover each other.

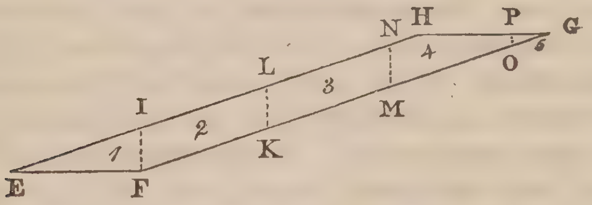
We shall now show that the figure F I H L K would exactly cover A N O C M. It has been proved that the angle E I F is equal to the angle A M B; but the angle E I F is equal to the angle I F K, and the angle A M B is equal to the angle M A N (42.); therefore the angle K F I is equal to the angle M A N; and since F K is equal to A M, and F I equal to A N, if the line F K be laid upon A M, F I will fall upon A N. But since the angle E I F is equal to the angle M A N, it is equal to the angle D N O; therefore the angle F I H, which is the supplement of E I F, is equal to the angle A N O, which is the supplement of D N O; and therefore I H will fall upon N O; but the angle G has been proved to be equal to the angle N O D; therefore the angle H, which is the supplement of G, is equal to the angle N O C,

which is the supplement of NOD; and therefore HL will lie upon OC; but since HG and DC are respectively equal to EF and AB, they are equal to each other; and since LG has been proved equal to DO, HL will be equal to OC; and therefore the point L will fall upon the point C; and since the angle HLK and the angle C are both right angles, the line LK must fall upon the line CM; and LK being equal to CM, K must fall upon M; thus figure FIHLK will exactly cover the figure ANOCM.

In fact, it will be apparent, when the pieces, marked 1, 2, and 3., in *fig.* 96., are considered, that it is only necessary to shift their position from right to left, so as to place FK upon EI, and KG upon HI, to transform the oblique parallelogram into a rectangle, identical with *fig.* 95.; the pieces marked 1, 2, and 3., taking the position assigned to them in *fig.* 95.

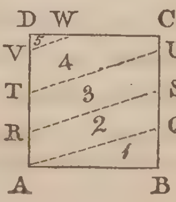
If the parallelogram be more oblique than that represented in *fig.* 96., it may be necessary to dissect it into smaller pieces, in order to convert it into a rectangle; the process, however, will, in principle, be the same. In *fig.* 97. is represented a more oblique rectangle, which

fig. 97.



is divided into five pieces; it will be easily perceived that, by shifting the position of these pieces, the parallelogram may be transformed into a rectangle, as represented in *fig.* 98.; and in the same manner every parallelogram, however oblique, may be transformed into a rectangle without changing its area; the base and height of such rectangle being equal to the base and height of the parallelogram.

fig. 98.



This important proposition, that the area of a parallelogram is dependent only on that of the rectangle under its base and altitude, and altogether independent of its shape, is one which would, upon attentive consideration, suggest itself to the understanding, almost without demonstration; in fact, the oblique parallelogram may be regarded merely as the surface of a rectangle, of the same height, thrown into a leaning position. If a rectangle be conceived to be formed by piling a number of thin plates one above another, it is evident that the extent of its area will not be altered if, by shifting the position of the plates, their edges are made to form an oblique parallelogram.

Let *fig.* 99. be conceived to represent the side view

fig. 99.

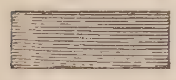


fig. 100.

of a pack of cards, so piled as to form a rectangle; if the position of the cards be changed, the rectangle may be converted into an oblique parallelogram, as represented in *fig.* 100. So long as the height and base remain the same, the parallelogram will be formed by the edges of the same cards, and must, therefore, have the same magnitude. If the height were less the number of cards must be less, and therefore the extent of area less in the same proportion. If the base were less the length of each card would be less, and, for that reason, the extent of area would be proportionally diminished.*

(209.) If the base and altitude of any parallelogram be expressed by numbers, its superficial magnitude, or area, will be expressed by the product of these numbers.

(210.) If two parallelograms have the same or equal bases, and the altitude of one be twice or thrice the altitude of the other, the area of the one will be twice or thrice the area of the other.

* In spirit this mode of demonstration is identical with that used for problems of quadrature in the higher geometry.

In general, when two rectangles, or parallelograms, have the same or equal bases, their areas will have the same numerical relation as their altitudes.

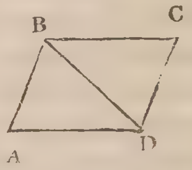
(211.) We shall call the relative magnitude of two lines, or surfaces, expressed numerically, their ratio; thus, if one line be 8 inches long, and another 10, their ratio is 8 to 10, or 4 to 5.

(212.) If two rectangles, or parallelograms, have the same or equal altitudes, their areas will have the same ratio as their bases; thus, if the base of one be twice or thrice the base of the other, the area of one will be also twice or thrice that of the other; or, if the base of the one be two thirds, or three fourths, of the base of the other, the area of the one will likewise be two thirds, or three fourths, of the area of the other.

(213.) A triangle may always be completed into a parallelogram by adding to it another equal triangle.

Let A B D, *fig.* 101., be the given triangle, and draw B C and D C, making the angle C B D equal to A D B, and the angle C D B equal to the angle A B D, the triangle C B D will then be, in all respects, equal to the triangle A B D; and since the angle C B D is equal to A D B, B C is parallel to A D; and since the angle C D B is equal to A B D, C D is parallel to A B; therefore the figure is a parallelogram.

fig. 101.



It is evident, that the base and altitude of the parallelogram thus formed, is equal to the base and altitude of the given triangle.

(214.) Hence it follows, that the area of a triangle is always equal to half the area of a parallelogram having the same base and altitude.

(215.) Hence, if the base and altitude of a triangle be expressed in numbers, its area will be also expressed numerically by half the product of these numbers (209.). The area of a triangle will therefore be formed by multiplying its base by its height, and taking half the product.

(216.) When triangles have the same or equal bases, their areas are in the same ratio as their heights.

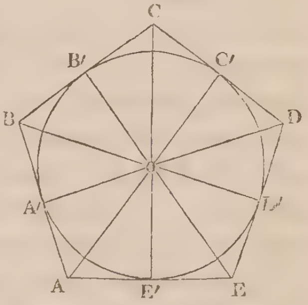
(217.) When triangles have the same or equal heights, their areas are in the same ratio as their bases.

(218.) In general, the areas of triangles are to each other in the same ratio as the products of their bases and altitudes.

(219.) All right-lined figures may be resolved into triangles by drawing diagonal lines; and, therefore, their areas may be determined by measuring the bases and altitudes of their component triangles, and thereby determining the areas of these several triangles.

(220.) If a polygon be such as to allow a circle to be inscribed in it, so that all the sides of the polygon shall be tangents to the circle, the area of the polygon will be equal to half the rectangle under the radius of the circle so inscribed, and the perimeter* of the polygon. For let A B C D E (*fig.* 102.) be the polygon: from the centre of the inscribed circle let lines be drawn to its several angles: these lines will resolve the area of the polygon into as many triangles as it has sides; and, considering the sides of the polygon as the bases of these triangles, respectively, their altitudes will be the radii of the inscribed circle O A′, O B′, O C′, &c. (192). Therefore the area of such triangle will be equal to half the rectangle under the radius of the circle and the side of the polygon; and the sum of all the areas of these triangles, or the area of the polygon, will be equal to half the rectangle under the radius of the circle, and the sum of all the sides, or the perimeter.

fig. 102.



(221.) Since all regular polygons admit of having a circle inscribed in them (192.), their areas will be

* The perimeter of a figure is the sum of all its sides, and corresponds to what is expressed by the word circumference in reference to the circle.

equal to half the rectangle under the radius of such circle and their perimeters.

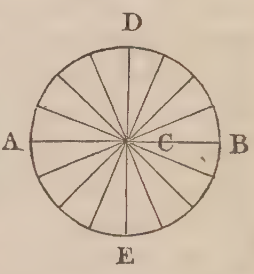
(222.) The area of a regular polygon will be equal to the rectangle under the radius of the inscribed circle, and the length of one of its sides multiplied by half the number of sides.

For since the sides are equal, the length of one side, multiplied by half their number, will be equal to half the perimeter.

(223.) The area of a circle is equal to half the rectangle under the radius and its circumference.

For if lines be drawn from the centre of the circle (*fig.* 103.), dividing the space round the centre into any number of equal angles, the area of the circle will be resolved into a corresponding number of equal sectors; and, if the chords of the arcs of these sectors be drawn, an inscribed polygon will be formed having these chords for its sides. If a circle be inscribed in this polygon, its radius will be a perpendicular from the centre on any of the chords. The area of the polygon will be equal to half the rectangle under the radius of the inscribed circle, and the perimeter of the polygon; but if the number of sectors into which the circle is divided be continually doubled by bisecting the angles (196.), the number of sides of the polygon will be continually increased, while their magnitude is diminished. The perimeter of the polygon will continually approach to coincidence with the circumference of the circle in which it is inscribed; and the radius of the circle inscribed in it will continually approach to equality with the radius of the circle circumscribed round it. As the two circles, and the polygon between them, approach without limit to absolute coincidence, the area of the polygon is continually equal to half the rectangle under the radius of the inscribed circle and its perimeter. Since this equality, therefore, is not disturbed by the

fig. 103.



varying state of the circles and polygon, it will still be maintained when that variation is carried to its limit, and these figures are brought to actual coincidence. In this case, however, the radius of the inscribed circle will be the radius of the circumscribed circle, and the perimeter of the polygon will be the circumference of the latter ; therefore, the area of the circle is equal to half the rectangle under its radius and circumference.

This may be made still more evident, if we actually cut two equal circles, like *fig.* 103., into the same number of small triangular gores ; and, instead of arranging them round centres, we arrange them, as here (*figs.* 104, 105.), with their bases placed in two straight

fig. 104.

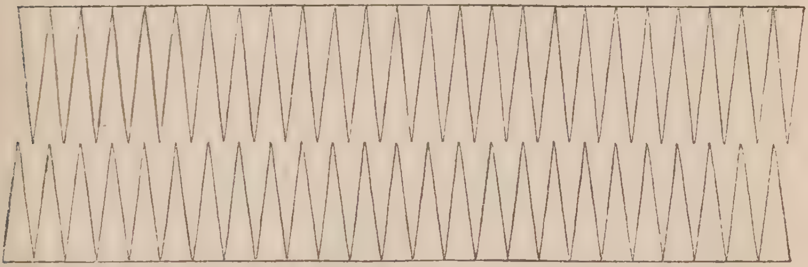


fig. 105.

lines parallel to one another, so as to present the appearance of the teeth of a saw. If they be moved towards one another, as here represented, so that the teeth of one may be inserted in the spaces between the teeth of the other, a parallelogram will be formed ; and if the arcs into which the circles are divided be exceedingly small, this parallelogram will be a rectangle, whose height will be the radius of the circle, and the base its circumference. It is plain, then, that the two added together, form a rectangle under the radius and circumference ; and, therefore, one of them alone will be equal to the rectangle under the radius, and half the circumference.

(224.) It has been already shown (102.) that the ratio of the circumference of a circle to its dia-

meter may be expressed with any degree of numerical precision which can be required. Hence, if the length of the radius of a circle be known, the length of its circumference can be immediately found: thus, twice the radius multiplied by 3·14 will be less than the circumference; and twice the radius multiplied by 3·15 will be greater than it. In the same manner, twice the radius multiplied by 3·141 will be less than the circumference; and twice the radius multiplied by 3·142 will be greater than it. Thus, to find the circumference, a number may be selected from the table, page 60., such as will give the circumference within the required limit of accuracy. This number, whatever it may be, which, being multiplied by the diameter, will give the circumference with the necessary precision, being frequently referred to in mathematics, is usually expressed by the Greek letter π. If r, then, be the radius of a circle, $r \times \pi$ will be its semi-circumference.

Since the area is equal to half the rectangle under the radius and circumference, it will be found by multiplying the radius by $r \times \pi$. But if we multiply r by r we obtain the square of the radius. Hence, when the radius of a circle is expressed by a number, its area will be immediately found by multiplying the square of that number by the number expressed by π.

Thus, for example, if the radius of a circle be 3 feet, its square will be 9; and if we require the area, we have only to multiply by 3·14, which gives 28·26 square feet for the area. If we multiply it by 3·15, we should get 28·35 square feet. The area, therefore, being between these, is obtained within a tenth of a square foot by this method.

If greater precision be required, the second numbers in the tables, page 60., may be taken. We should then multiply 9 by 3·141, and we should find 28·269 square feet; and by multiplying it by 3·142, which would give 28·278 square feet, we should thus obtain the true area within the hundredth part of a square foot, and so on.

(225.) The following method, which is, in fact, equivalent to the principle of the table in page 60., and which will give the area within a very minute fraction of the square of the radius, may be used with convenience.

Multiply the square of the radius by 355, and divide the product by 113. Thus, with a radius of 3 feet, as before, we multiply 355 by 9, by which we obtain 3195, which, divided by 113, will give 28·274 square feet; and still greater accuracy might be obtained, if the division by 113 were continued farther.

(226.) It appears, therefore, that the area of the circle has to the square of its radius the ratio of 355 to 113, very nearly.

(227.) A square circumscribed round a circle is four times the square of its radius; the area of the circle will have to such a square the ratio of 355 to four times 113, or of 355 to 452, very nearly.

(228.) The area of a circle may therefore always be obtained by multiplying the square of its diameter by 355, and dividing the product by 452.

(229.) Since the area of all circles have the same ratio to the squares of their diameters, the areas of different circles are to each other in the same ratio as the squares of their diameters.

(230.) Also, since the circumferences of different circles have the same ratio to their diameters, the circumference of different circles will be in the ratio of their diameters.

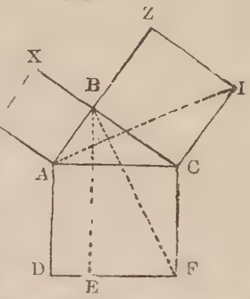
(231.) Hence, if a series of circles have diameters expressed by the successive whole numbers, 1, 2, 3, 4, 5, &c., their circumferences will be proportional to the same numbers; the circumference of the second, third, fourth, fifth, &c. being twice, three times, four times, &c. that of the first: their areas, being as the squares of their diameters, will be expressed by the numbers 1, 4, 9, 16, 25, &c.

(232.) In a right-angled triangle — if squares be constructed upon the three sides, that which is constructed

on the side opposite to the right angle will be equal to the other two added together.

On the sides A B, A C, and B C, (*fig.* 106.) describe the squares A X, A F, and B I, and draw B E parallel to either C F or A D, and join B F and A I.

fig. 106.



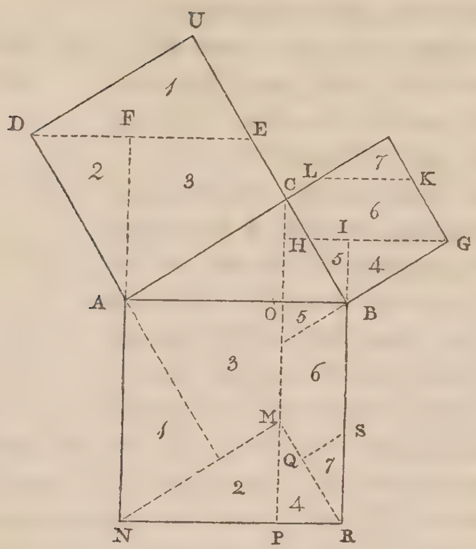
Because the angles I C B and A C F are equal, if B C A be added to both, the angles I C A and B C F are equal, and the sides I C, C A, are equal to the sides B C, C F; therefore the triangles I C A and B C F are equal (59). But A Z is parallel to C I; therefore the parallelogram C Z is double of the triangle I C A, as they are upon the same base C I, and between the same parallels (214.); and the parallelogram C E is double of the triangle B C F, as they are upon the same base C F, and between the same parallels (214.); therefore, the parallelograms C Z and C E being double of the equal triangles I C A and B C F, are equal to one another. In the same manner it can be demonstrated, that A X and A E are equal; therefore the whole D A C F is equal to the sum of C Z and A X.

(233.) It may not be uninteresting, in a proposition of such extreme importance as the preceding, and so conspicuous for its beauty, to show how, by actual dissection, the square on the side opposite to the right angle, may be made to cover the squares of the two sides which form it.

From D and G (*fig.* 107.) draw D E and G H parallel to A B, and produce N A and R B to meet these parallels at F and I; take C M equal to A N or B R, and draw M N and M R, which will be parallel to C A and C B; produce D A and G B, as in the figure; take G K equal to B H and draw K L parallel to A B, and take R S equal to K L, and draw S Q parallel to B G.

The square on the side opposite the right angle, is now divided into seven pieces; and the squares on the sides which form it are likewise divided into seven

fig. 107.

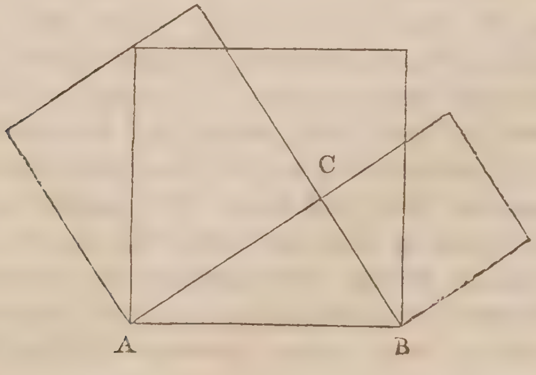


pieces; it admits of easy proof that each of the pieces into which the great square is divided will exactly cover the piece marked with the same number in the lesser squares; we shall, however, leave the complete investigation of this to the student.

The same proposition may also be demonstrated in the following manner: —

Let the three squares be constructed on the same side of the base (*fig.* 108.); the triangle ACB thus forms a part of the great square; let it be supposed to

fig. 108.



be removed from its present position and placed on the upper side of the square in the position D E F (*fig.* 109.). The great square is now converted into the six-sided

fig. 109.

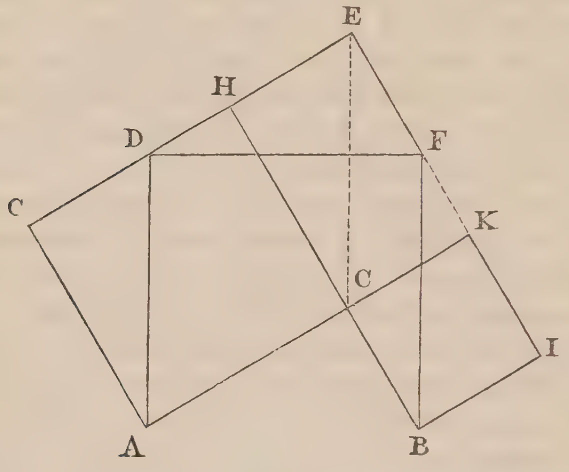
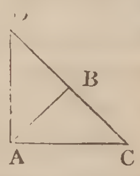


figure A D E F B C; if the line E C be drawn this figure will be resolved into two oblique parallelograms; the first, A D E C, standing on the base A C, and the other, B F E C, standing on the base B C. But by what has been already proved (208.), A D E C may be dissected so as to cover the square A C H C, and B F E C may be dissected so as to cover the square C K I B; therefore the two parallelograms, or the great square to which they are equal, will exactly cover the squares on the sides which form the right angle.

(234.) If the squares of two sides of a triangle be equal, taken together, to the square of the third side, the angle opposite the third side must be a right angle.

fig. 110.



Let the squares of A B and B C, taken together, be equal to the square of A C (*fig.* 110.) then the angle A B C will be a right angle.

For from the point B draw B D perpendicular to one of the sides A B and equal to the other B C, and join A D.

The square of A D is equal to the squares of A B

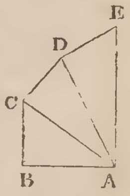
and B D (232.), or to the squares of A B and B C, B C being equal to B D. But the squares of A B and B C are together equal to the square of A C; therefore the squares of A D and A C are equal, and therefore the lines themselves are equal; but also D B and B C are equal, and the side A B is common to both triangles; therefore the triangles A B C and A B D are in all respects equal, and therefore the angle A B C is equal to the angle A B D. But A B D is a right angle, therefore A B C is also a right angle.

(235.) Hence this peculiar relation among the squares of the sides is a distinguishing character of a right-angled triangle. That it is a property of a right-angled triangle, appears by (232.), and that it is a property of no other triangle is established by (234.).

(236.) This principle furnishes a method of adding together two or more squares, so as to obtain a square equal to their sum.[1]

Let several lines be given to find a line whose square is equal to the sum of their several squares.

fig. 111.

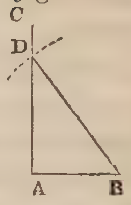


Draw two lines A B and B C (*fig.* 111.) at right angles, and equal to the first two of the given lines, and draw A C. Draw C D equal to the third of the given lines, and perpendicular to A C, and draw A D. Draw D E equal to the fourth, and perpendicular to A D, and draw A E, and so on; the square of the line A E will be equal to the sum of the squares of A B, B C, C D, D E, which are respectively equal to the given lines.

For the sum of the squares of A B and B C is equal to the square of A C. The sum of the squares of A C and C D, or the sum of the squares of A B, B C, C D, is equal to the square of A D, and so on. The sum of the squares of all the lines is equal to the square of A E.

(237.) A square may also be formed which shall be equal to the difference between the squares of two given lines.

fig. 112.



Through one extremity A (*fig.* 112.) of the lesser line A B, draw an indefinite perpendicular A C; from the other extremity B inflect on A C with the compasses, a line equal to the greater of the two given lines, which is always possible since the line so inflected is greater than B A, which is the shortest line which can be drawn from B to A C. The square of A D will be equal to the difference of the squares of B D and B A, or of the given lines.

(238.) If the two sides which form the right angle of a right-angled triangle be expressed by numbers, the number which will express the square of the third side will be found by adding together the numbers expressing the squares of the other sides.

Hence the number expressing the side opposite the right angle may be found by adding together the squares of the sides which form it, and taking the square root of their sum.

(239.) In the same manner, if the side opposite the right angle be given in numbers, the third side may be found; for, if from the square of the side opposite the right angle, the square of the given side be substracted, the remainder will be the square of the third side, and its square root will be the third side itself. Therefore, to find the third side of a right-angled triangle, when the side opposite the right angle and another side are given in numbers, it is only necessary to take the square of the lesser given side from the square of the greater, and the square root of the remainder will be the third side.

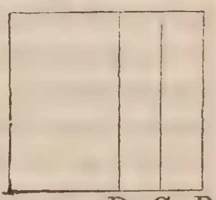
(240.) If the three sides of a triangle be expressed by numbers, it may be known whether it be a right-angled triangle or not, by comparing the square of the greatest of the three sides with the sum of the squares of the other two: if the latter be equal to the former the triangle will be right-angled, otherwise not.

(241.) If a line be divided into several parts, the square of the line will be equal to the several rectangles

under the line, and each of the parts into which it is divided.

Let the line A B (*fig.* 113.) be divided into three parts, at C and D, and let a square be described upon it; and, from the points of division C and D let perpendiculars be drawn: these perpendiculars will evidently divide the square into three rectangles under the line A B, and its three several parts.

fig. 113.

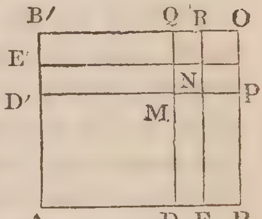


In the same manner it may be shown that whatever be the number of parts into which the line is divided, the square of the line is equal to the several rectangles under the line and each of its parts.

(242.) If a line be divided into two or more parts, the square of the whole line is equal to the squares of the several parts together with twice the rectangles under every pair of parts.

Let the line A B (*fig.* 114.) be divided into three parts at D and E, and let a square be constructed upon it, and divide the side A B′ at D′ and E′ into similar parts; from D and E let perpendiculars be drawn to A B, and from D′ and E′ let perpendiculars be drawn to A B′.

fig. 114.



Since A D′ is equal to A D, A D′ M D is the square of A D; and since D E is equal to D′ E′, the parallelogram M N is the square of D E, its sides being respectively equal to D E and D′ E′; and in like manner the parallelogram N O may be proved to be the square of E B.

The rectangles E M and E′M are rectangles under A D and D E; the rectangles E P and E′Q are rectangles under A D and E B; and the rectangles N P and N Q are the rectangles under D E and E B: thus the whole square of A B is resolved into the squares of the three parts, and twice the rectangles under eacn pair of these parts; and in the same manner, if the line were

divided into any other number of parts, it might be proved that the square of the whole line would be equal to the squares of the parts, and twice the rectangle under each pair of them.

Hence, if a line be divided into two equal parts, the square of the line is equal to the squares of the two parts, and twice the rectangle under them.

(243.) If the two parts into which a line is divided are equal, the rectangle under them is the square of half the line; therefore the square of the whole line is equal to four times the square of half the line.

CHAP. X.

OF SIMILAR FIGURES.

(244.) Two geometrical figures which have the same shape or form, but are constructed on a different scale, are said to be similar figures.

The sides and all the corresponding dimensions of such figures must have the same proportion one to the other, and their corresponding angles must be equal; thus, if any one side of one of the figures be double or triple the corresponding side in the other figure, then every side in the one must be double or triple the corresponding side in the other; and the angle formed by each pair of sides in the one must be equal to the angle formed by the corresponding sides in the other.

This important relation, constituting the similarity of geometrical figures, though it may not be perceived with clearness or facility when expressed in an abstracted and general form, is the relation of magnitudes with which, perhaps, we are most familiar in the arts and in the ordinary business of life. The delineation of maps and plans consists in expressing on a small scale, but without disturbing their proportions, the shape of tracts of country: in other words, it consists in drawing a *similar figure* with shorter sides.

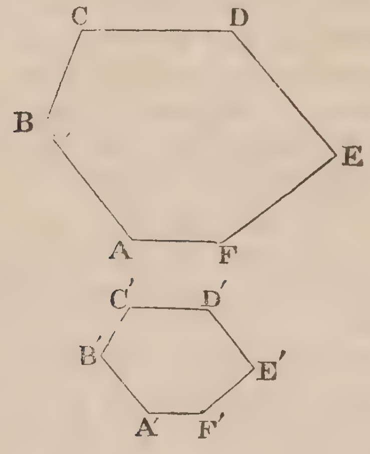
In like manner, the representation of all objects in painting, whether they be landscapes, portraits, or, in fine, representations of any natural or artificial objects, consists, only, in drawing figures similar to the outlines of these objects on a reduced scale.

(245.) The precise conditions under which two geometrical figures will be similar are the following: 1st., that they shall have the same number of angles; 2d., that these angles shall be respectively equal, each to

each; 3d., that the sides containing the angles which are equal, shall have to each other the same proportion.

Thus, if ABCDEF (*fig.* 115.) be said to be similar to A′B′C′D′E′F′,

fig. 115.



it is meant that the number of angles being the same in both, the angle A shall be equal to A′, B to B′, C to C′, D to D′, E to E′, and F to F′; also, that whatever ratio the side AB shall have to the side A′B′, the same ratio shall BC have to B′C′, CD to C′D′, DE to D′E′, EF to E′F′, and FA to F′A′.

Thus, if AB be twice A′B′, then BC will be twice B′C′, and so on.

(246.) In triangles the equality of the angles necessarily infers the other condition of similitude, viz. the proportionality of the sides; and, *vice versâ*, the proportionality of the sides infers the equality of the angles. Thus, if in two triangles, either of the conditions of similarity be fulfilled, the other condition must also be fulfilled. This is a peculiarity of triangles; it belongs to no other right line figure, as will be evident upon the slightest consideration; since, if it did, the proportion of the sides being given, the angles would be unalterable. Now it has been already proved, that in a figure of four or more sides, jointed with pivots at the angles, the angles may be altered in an infinite variety of ways. In fact, the

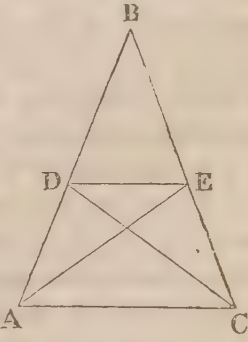
characteristic property of triangles here noticed is dependent upon, and is an extension of, the property already proved (62.), in virtue of which two triangles are equal in all respects, if their sides be mutually equal.

To derive from first principles this property of triangles in its most general form has been attended with some difficulty, as has indeed been the case with every general proposition arising out of our ideas of ratio or proportion. Mathematicians have differed much as to the definition of these terms themselves, owing to the difficulty of including those particular cases which, like the diameter and circumference of a circle, cannot be precisely expressed by definite numbers, and which have therefore been called incommensurable quantities.

However useful disquisitions of this kind may be to those who prosecute the study of geometry chiefly as an intellectual exercise, they are attended with little benefit either to those who on the one hand cultivate the science merely with a view to its application in the arts, or to those who on the other hand intend to penetrate to the more abstruse departments of mathematics: — for the one class of students more simple and less abstract views will be sufficient, and the latter will find their views of this question satisfactorily cleared up as they ascend to the higher branches of analysis.

(247.) If in the triangle A B C (*fig.* 116.), parts B D and B E be taken on the sides B A and B C which shall be proportional to those sides, the line D E will be parallel to the base A C.

fig. 116.



For let the lines D C and E A be drawn, the two triangles B D C and B A C having for their bases the lines B D and B A, and having their common vertex at C, have the same height, and therefore their areas will be in the same ratio as their bases (217.); that is, their areas will be as B D to B A.

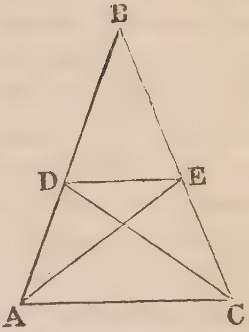
In the same manner the triangles B A E and B A C, considering the lines B E and B C as their bases, have a common vertex at A, and therefore have the same altitude. Their areas are therefore as their bases, that is, as B E to B C (217). Thus it appears that the areas of the triangles B D C and B E A have respectively, to the area of the given triangle A B C, the ratio of the parts B D and B E cut off from the sides to the whole sides B A and B C. But these parts cut off are proportional to the sides, that is, each of them has the same ratio to the side from which it is cut; and therefore the areas of each of the triangles B D C and B E A have the same ratio to the area of the whole triangle, and are therefore equal.

This conclusion will be apprehended more easily and clearly if it be stated in a less general form: thus if B D and B E be respectively half of B A and B C, then the triangles B A E and B C D will be respectively half of the whole triangle, and will therefore be equal. In the same manner if B D and B E were respectively a third or a fourth, or two thirds or three fourths of B A and B C, the areas of the triangles B D C and B E A would be respectively a third or a fourth, or two thirds or three fourths of the whole triangle, and would therefore be equal.

Since then the areas B D C and B E A are equal, if we take away the area B D E from both, the remainders A D E and C E D will be equal: now these two triangles have D E as a common base; and since their areas are equal, the perpendicular from their vertices A and C to this common base D E must be equal; therefore the points A and C are equally distant from the line D E, and consequently the line D E must be parallel to A C.

Hence any line which, like D E, cuts off parts from the sides of a triangle proportional to these sides will be parallel to the base.

fig. 117.



(248.) On the other hand, if a line D E (*fig.* 117.) be drawn parallel to the base of a triangle, it will cut off parts B D and B E of the sides which are proportional to the whole sides.

This may be demonstrated by a process similar in all respects to the last.

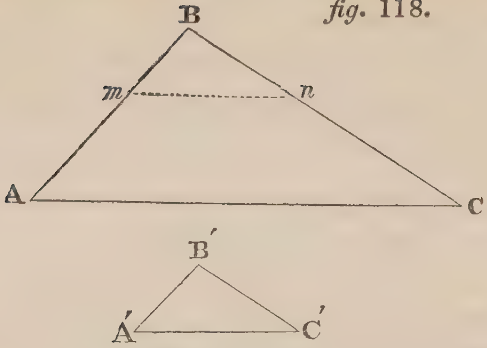
The lines D C and E A being drawn, the triangles which have D E as their common base, and their vertices at A and C, will have equal areas; because they have the same base, and having their vertices in a parallel to that base, they have equal altitudes.

To these equal areas let the area B D E be added, and then the areas of the triangles B A E and B C D will be equal; these equal areas, therefore, will bear the same ratio to the area of the whole triangle. But the triangles B D C and B A C having a common vertex C, have the same altitude; there areas will therefore be as their bases B D and B A (217.); and for a like reason the areas of B E A and B C A will be as their bases B E and B C. Since, therefore, the equal areas B D C and B E A have the same proportion to the whole area, the parts B D and B E will have the same proportion to the sides B A and B C.

(249.) We are now prepared to demonstrate the property of triangles in virtue of which either of the two conditions of similitude infers the other.

If two triangles, A B C and A′B′C′ (*fig.* 118.), be respectively equiangular, the angles marked by the same letters being equal, their corresponding sides will have the same proportion each to each; for let the vertex of the angle B′ be placed upon the vertex of the angle B; and let the sides of the angle B′ lie upon those of the angle B, which is equal to it, so that the point A′ shall fall at *m* upon the side A B, and the point C′ at *n* upon the side C B: since the angle A′ is equal to the angle A,

fig. 118.



the line $m\,n$ will be parallel to the side AC; and therefore (248.) the line Bm, or the side B′A′, shall have to the side BA the same ratio as the line Bn, or the side B′C′, has to the side BC.

In the same manner, by placing the angle A′ upon the angle A, it may be shown that the side A′C′ has to the side AC the same ratio as the side A′B′ has to the side AB.

(250.) A peculiar notation is used in Arithmetic to express ratios, which is transferred also to Geometry. In Arithmetic, the sign : between two numbers expresses their ratio; and, in like manner, in Geometry the same sign between the letters expressing two lines expresses their ratio.

Thus, if the letters a, b, c express respectively the sides of the triangle opposite to the angles A, B, C, and also a', b', c' express the sides of the other triangle opposite to the angles A′, B′, C′ respectively, then the ratios of the corresponding sides will be expressed by $a : a'$, $b : b'$, and $c : c'$.

(251.) In Arithmetic, also, the equality of two quantities is expressed by the sign $=$ placed between them; and the same sign is transferred to Geometry. Thus, the angle A being equal to the angle A′, their equality is expressed thus, $A = A'$.

The same sign of equality is extended, both in Arithmetic and Geometry, to ratios. Thus it was proved that the ratios of each pair of corresponding sides in the

two triangles were equal. This would be expressed, by the notation just explained, thus:—

$$a : a' = b : b' = c : c'.$$

And the proposition demonstrated in (250.) would be expressed thus:—

$$\text{If } A = A', B = B', \text{ and } C = C',$$
$$\text{then } a : a' = b : b' = c : c'.$$

In other words, the equality of the angles infers the proportionality of the sides.

(252.) On the other hand, the proportionality of the sides may be proved to infer the equality of the angles: *i. e.*

$$\text{If } a : a' = b : b' = c : c',$$
$$\text{then, } A = A', B = B', \text{ and } C = C'.$$

For on the sides a and c, that is, on BC and BA, take parts Bn and Bm equal to B′C′ and B′A′ respectively; now, since $a : a' = c : c'$, the line $m\,n$ cuts proportional parts from the sides of the triangle ABC, and therefore $m\,n$ is parallel to AC (247.); and therefore the angle B$m\,n$ is equal to the angle A, and the angle B$n\,m$ is equal to the angle C. The angles therefore of the triangle B$m\,n$ being respectively equal to those of the triangle BAC, the sides of these triangles will be proportional (249.); therefore AC : $m\,n$ = AB : B$m = c : c'$.

But also AC : A′C′ $= c : c'$, therefore AC : A′C′ = AC : $m\,n$. In other words, the side AC has the same ratio to A′C′ as it has to $m\,n$, and therefore A′C′ must be equal to $m\,n$.

The three sides of the triangle therefore, B$m\,n$, are respectively equal to those of B′A′C′, and therefore the angles are equal (62.); that is, the angle A′ is equal to the angle B$m\,n$, the angle C′ is equal to the angle B$n\,m$, and the angle B′ is equal to the angle B: but it has been proved that the angle B$m\,n$ is equal to the angle A, and that the angle B$n\,m$ is equal to the angle C; therefore the angle A′ is equal to the angle A, and the angle C′ is equal to the angle C. So that the

two triangles ABC and $A'B'C'$, which have their sides proportional, have their angles respectively equal.

(253.) If two triangles have an angle in one equal to an angle in the other, and the sides which include that angle proportional each to each, the triangles will be similar.

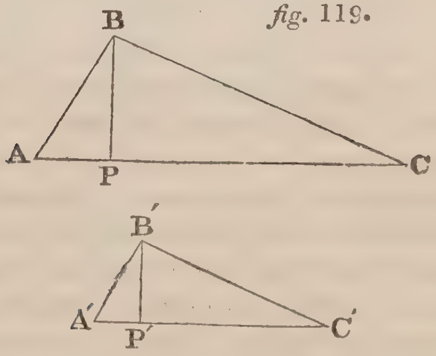
That is, if $B = B'$ and $a : a' = c : c'$, then will $A = A'$ and $C = C'$.

For on the sides of the angle B take parts $B\,m$ and $B\,n$ equal to $B'A'$ and $B'C'$; it may be proved as before, that in this case $m\,n$ must be parallel to AC, that the triangle $B\,m\,n$ will be similar to BAC, and that it will be in all respects equal to $B'A'C'$. Therefore the triangle $B'A'C'$ will be similar to the triangle BAC.

(254.) If two triangles be similar, perpendiculars drawn from angles on the opposite sides will divide them into similar right-angled triangles, and these perpendiculars will, themselves, be proportional to any two corresponding sides of the triangles.

Let ABC and $A'B'C'$ (*fig.* 119.) be the two triangles;

fig. 119.



and let BP and $B'P'$ be the two perpendiculars drawn from the equal angles B and B' on the opposite sides. The triangles BAP and $B'A'P'$ will then be similar, as will also be the triangles BPC and $B'P'C'$.

For since the given triangles are similar, the angles A and A' are equal; and the angles BPA and $B'P'A'$ are equal, being right; therefore the triangles BPA

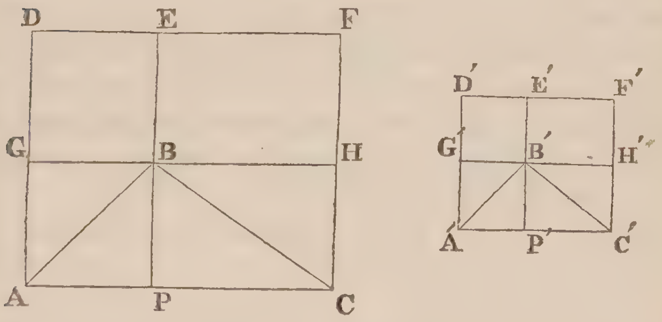
and B′P′A′ are mutually equiangular, and are therefore similar. And in the same manner it may be shown that the triangles BPC and B′P′C′ are similar.

The perpendicular BP has to B′P′ the same ratio as BA has to B′A′, being corresponding sides of the similar right-angled triangles; that is, the perpendiculars are proportional to the corresponding sides of the given triangles.

(255.) The areas of similar triangles are proportional to the squares of their corresponding sides.

Let ABC and A′B′C′ (*fig.* 120.) be similar tri-

fig. 120.



angles, and let squares be constructed upon the sides AC and A′C′; and through B and B′ let lines be drawn perpendicular to AC and A′C′, and therefore parallel to the sides of the squares. Through B and B′ also draw GH and G′H′ parallel to AC and A′C′.

The areas of the triangles being the halves of the rectangles under their bases and altitudes are proportional to these rectangles; that is, to the rectangles AGHC and A′G′H′C′. But the rectangle AGHC has to the square ADFC the ratio of their heights PB and PE, since they have the same base AC; therefore the square constructed on AC has to the rectangle under the base and altitude of the triangle the ratio of the base to the altitude; and in the same manner it may be shown that the square constructed on the base A′C′ has to the rectangle under the base and altitude of the

other triangle the same ratio as the base to the altitude.

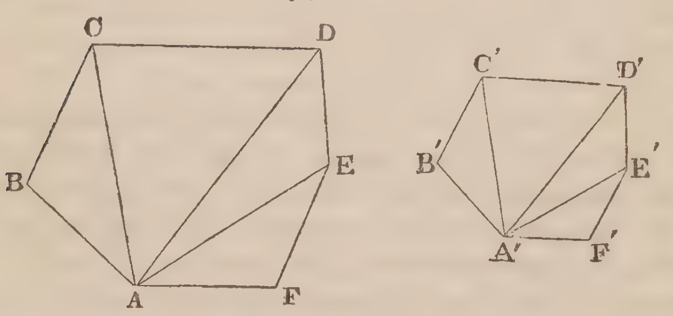
But it has been already shown (254.) that, in similar triangles, corresponding altitudes or perpendiculars are proportional to corresponding sides; therefore the squares of the corresponding sides have the same ratio to twice the areas of the triangles, and therefore have the same ratio to the areas themselves, and therefore the areas are as the squares of the corresponding sides.

Thus, if in two similar triangles the sides of one are twice, three times, or four times the corresponding sides of the other, the area of the one will be four times, nine times, or sixteen times the area of the other; the areas being always proportional to the squares of the numbers which express the corresponding sides.

(256.) If two right-lined figures be similar, diagonals drawn from corresponding angles will resolve them into systems of triangles which will be similar each to each.

Let ABCDEF (*fig.* 121.) and A′B′C′D′E′F′ be

fig. 121.



similar figures, the angles expressed by the same letters being equal. From the angles A and A′ draw in each three diagonals to the angles C, D, E, in the one, and C′, D′, E′, in the other.

Since AB : BC = A′B′ : B′C′, and the angle B is equal to the angle B′, which is an immediate consequence of the similarity of the figures, the triangle ABC must be similar to the triangle A′ B′ C′ (253.), therefore the angle BCA must be equal to the angle B′C′A′; and if these be taken away from the angles BCD and

B′C′D′, the remaining angles ACD and A′C′D′ must be equal. Also in consequence of the similarity of the triangles ABC and A′B′C′, AC : A′C′ = BC : B′C′; but in consequence of the similarity of the given figures BC : B′C′ = CD : C′D′, therefore AC : A′C′ = CD : C′D′; and since the angle ACD is equal to the angle A′C′D′, the triangles ACD and A′C′D′ will be similar (253.); and in the same way every pair of triangles formed by corresponding diagonals may be proved to be similar.

(257.) It is evident that any two corresponding diagonals in such figures will be proportional to two corresponding sides.

(258.) The areas of any two corresponding component triangles will be as the squares of corresponding sides of the figures, since such sides are always corresponding sides of such triangles. Thus, the areas of every pair of corresponding triangles will be in the same ratio, since every pair of corresponding sides in the figures are in the same ratio.

(259.) Since the areas of every pair of corresponding triangles are as the squares of corresponding sides of the figures, the areas of all the triangles taken together, that is, the areas of the figures themselves, are in the same ratio; and thus we arrive at the conclusion that all similar figures, as well as similar triangles, have their areas proportional to the squares of their corresponding sides.

It will therefore be apparent that in varying the scale of a figure preserving its form, its superficial dimensions change much more considerably than its linear dimensions.

If we double its linear dimensions, we quadruple its superficial dimensions; if we increase its linear dimensions in a three-fold or four-fold ratio, we increase its superficial dimensions in a nine-fold or sixteen-fold proportion, and so on.

From what has been proved respecting circles (229.) (230.) it will be perceived that they, in all respects, participate the qualities of similar figures.

The perimeters of similar polygons being composed

of sides which are each to each in the same ratio, will be themselves in that ratio; thus it is evident that if each side in the one be twice the corresponding side in the other, the perimeter of the one, or the sum of all its sides, will be double the perimeter of the other, or the sum of all its sides.

The perimeters of similar polygons, therefore, have the same property as the circumferences of circles; they are proportional to any two corresponding lines in the figure. Thus as the circumferences of circles are in the same proportion as their diameters, the perimeters of similar polygons are in the same ratio as any two corresponding diagonals.

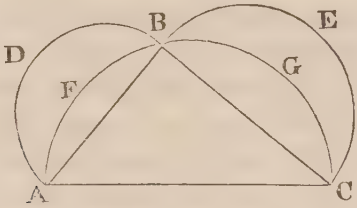
Also as the areas of circles are proportional to the squares of their diameters, so the areas of similar polygons are proportional to the squares of their corresponding diagonals.

(260.) It has been proved that, if squares be constructed on the three sides of a right-angled triangle, those which are constructed on the sides forming the right angle are equal, taken together, to the square constructed on the side opposite the right angle. But since any similar figures whatever constructed on three sides of the right-angled triangle, in which those three sides shall have corresponding positions, will be proportional to the squares of those sides, it follows that the above property extends to all similar figures; and therefore that, if any three similar figures shall have the three sides of a right-angled triangle for their corresponding sides, the areas of the two figures on the sides forming the right angle will be equal, taken together, to the figure constructed on the side opposite to it.

Since circles are as the squares of their diameters, this property may also be extended to circles; so that, if three circles be described having for their diameters the three sides of a right-angled triangle, the areas of those whose diameters form the right angle will, taken together, be equal to the area of a circle whose diameter is opposite the right angle.

(261.) Let A B C (*fig.* 122.) be a right-angled triangle, the angle B being the right angle. Let semicircles be

fig. 122.



described on B C and B A, and let a semicircle also be described on A C: this last semicircle must pass through the vertex of the right angle B; since the area of the semicircle A F G C is equal to the areas of the semicircles A D B and B E C, taken together, if the segments A F B and B G C be taken from both, the remainders will be equal; therefore the areas of the crescents D F and E G, taken together, will be equal to the area of the triangle A B C.

(262.) It has been proved in arithmetic, that if four numbers be proportional, the first to the second as the third to the fourth, the product of the means will be equal to the product of the extremes; the means being the second and third, and the extremes the first and fourth.*

Since the area of a rectangle is expressed by the product of the numbers which express its sides, we may at once transfer the above principle to geometry, announcing it as follows: —

If four lines be proportional, the first to the second, as the third to the fourth, then the rectangle under the means will be equal to the rectangle under the extremes.

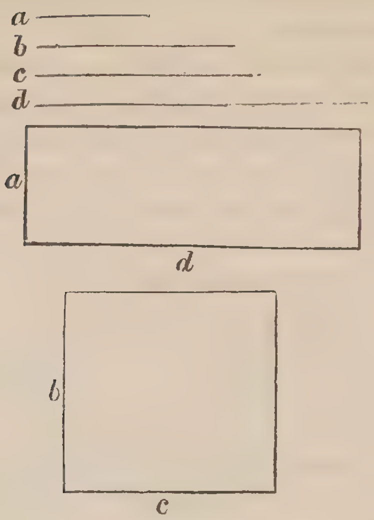
Thus let a, b, c, d, be four lines, and let

$$a : b = c : d.$$

Then the rectangle under a and d will be equal to the rectangle under b and c.

* *Arithmetic* (Cab. Cyc.), p. 375.

fig. 123.



(263.) If the length of three lines a, b, and c, be expressed numerically, a fourth proportional to them may be found by multiplying together the numbers expressing the second and third, and dividing the product by the number expressing the first.

Thus, if a, b, and c be given to find d; multiply b by c and divide the product by a, and the quotient will be d.

(264.) When three magnitudes are so related that the ratio of the first to the second is equal to the ratio of the second to the third, they are said to be in *continued proportion;* and the third is said to be a third proportional to the first and second, and the second a mean proportional between the first and third.

Thus, if $$a : b = b : c;$$

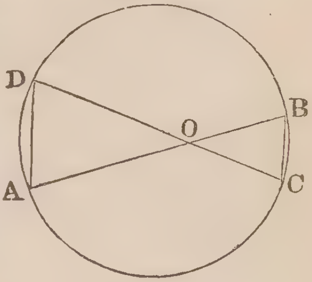
then c is a third proportional to a and b, and b is a mean proportional between a and c.

(265.) Since three continued proportionals may be considered equivalent to four proportionals having equal means, the square of the mean in three proportionals will be equal to the rectangle under the extremes, since the square

of the mean is in fact the rectangle under the two equal means of the three continued proportionals regarded as four proportionals with equal means.

(266.) If two chords intersect each other in a circle, the rectangle under the segments of the one will be equal to the rectangle under the segments of the other.

fig. 124.



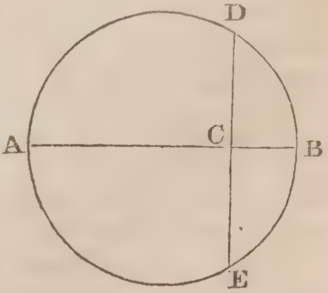
Let A B and C D be two such chords intersecting at O; draw the lines B C and D A; the angles A D C and A B C standing on the same arc of the circle are equal (110.), and the angles at O in the two triangles are also equal (20.); therefore the triangles A D O and C B O are mutually equiangular (57.), and are therefore similar. Hence (249.)

$$AO : DO = CO : BO.$$

The rectangle under the means will then be equal to the rectangle under the extremes; that is, the rectangle under A O and B O is equal to the rectangle under D O and C O.

It is evident that the same will be true for any number of chords intersecting in the same point; the rectangle under the segments of each of them will have the same magnitude.

fig. 125.



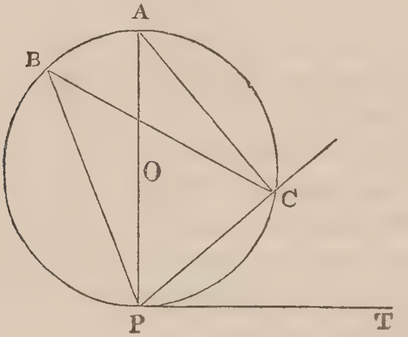
(267.) If A B (*fig.* 125.) be the diameter of a circle, a perpendicular to it from any point C, meeting the circle at D, will be a mean proportional between the segments A C and C B of the diameter.

For it has been already proved (116.) that if D E be perpendicular to the diameter A B, it will be bisected by the diameter; but since the rectangle under A C and C B

is equal to the rectangle under D C and C E, and D C is equal to C E, the rectangle under A C and C B will be equal to the square of D C; therefore D C will be a mean proportional between A C and C B. (265.)

(268.) If from the same point P (*fig.* 126.) on the

fig. 126.



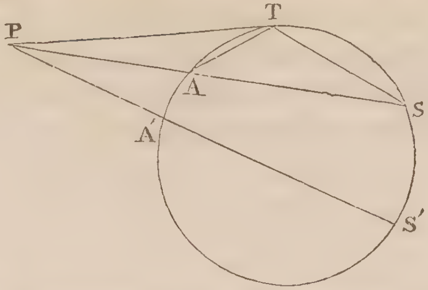
circumference of a circle, a tangent P T, and a chord P C, be drawn, the angle C P T, formed by these lines, will be equal to an angle contained in the segment of the circle P A C which lies on the other side of the chord.

For from P, through the centre O, draw the diameter P O A, and draw A C; the angle P A C is equal to all the other angles, such as P B C, in the same segment; and it is therefore necessary only to prove that the angle C P T is equal to the angle P A C.

The angle A P T is a right angle (83.), and therefore A P C and C P T are together equal to 90°; also the angle A C P is a right angle (112.), and therefore the angles C A P and C P A are together equal to a right angle (52.). Since the angle C A P, together with C P A, makes up 90°, and also C P T, together with the same angle C P A, is 90°; the angle C P T is equal to the angle C A P, and therefore equal to any angle in the same segment (110).

(269.) If from the same point P (*fig.* 127.) outside a circle a tangent P T and a secant P A S be drawn, the

fig. 127.



square of the tangent P T will be equal to the rectangle under P A and P S.

For let T A and T S be drawn: the angle P T A will be equal to the angle S (268.), and the angle P is common to the two triangles P A T and P T S; therefore the triangles are equiangular (57.), and are therefore similar; therefore their corresponding sides are proportional (249.): hence

$$PA : PT = PT : PS.$$

That is, PT is a mean proportional between PA and PS, and therefore the square of PT is equal to the rectangle under PA and PS.

(270.) Since this will be equally true of all secants drawn from the same point P, it follows that the rectangle under the corresponding lines for each secant are equal. Thus, if PA′S′ be drawn, the rectangle under PA′ and P S′ may in the same manner be proved to be equal to the square of PT, and is therefore equal to the rectangle under PA and PS, and the same will be true of all secants drawn from the same point P.

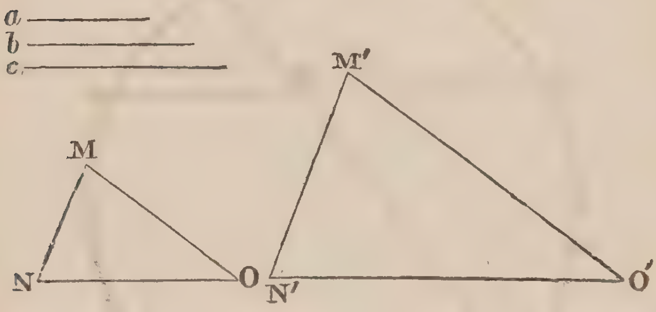
(271.) To find by geometrical construction a fourth proportional to three lines, is equivalent to the problem to construct upon a given right line a rectangle equal to a given rectangle; for the fourth proportional will be the height of a rectangle formed on the first of the three given lines whose area is equal to the rectangle under the second and third. Or the question may be stated thus, — The means and one extreme of four proportionals are given, and the other extreme is sought. Since the rectangle under the means is equal to the

rectangle under the extremes, the problem is to construct upon the given extreme a rectangle whose area shall be equal to the rectangle under the means.

The principles of geometry which have been already explained present many methods of doing this.

I. Let a (*fig.* 128.) be the given extreme, and b and c

fig. 128.



the given means. Draw two lines M N and M O equal respectively to a and b, and draw N O so as to form a triangle; also draw M′N′ equal to c, and on M′N′ construct a triangle having its angles equal to those of M N O; the two triangles being respectively equiangular, will be similar, and therefore their sides will be proportional: hence

$$\mathrm{MN} : \mathrm{MO} = \mathrm{M'N'} : \mathrm{M'O'},$$
$$\text{or } a : b = c : \mathrm{M'O'};$$

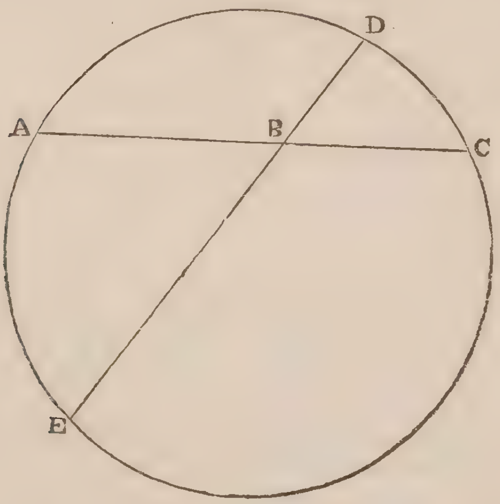
M′O′ is therefore the fourth proportional which is sought, and the rectangle under M′O′ and a will be equal to the rectangle under b and c.

It will be perceived that the spirit of this solution consists in making the given extreme and one of the means two sides of a triangle, and in constructing a similar triangle of which the other mean and the sought extreme shall be corresponding sides. Although the other varieties of solution for this problem are apparently different from the present, yet if carefully considered they will be found to be identical with it; the only difference being in the method by which the two similar triangles are constructed.

II. The same problem may be solved otherwise, thus: —

Let the given means or lines equal to them be placed so as to form one straight line, so that A B (*fig.* 129.)

fig. 129.



shall be equal to b, and B C to c; from B draw B D equal to a, and making any angle with A C; through the points A, D, C, describe a circle (121.); produce the line D B until it meet the circle at the opposite side E; B E will then be the fourth proportional sought.

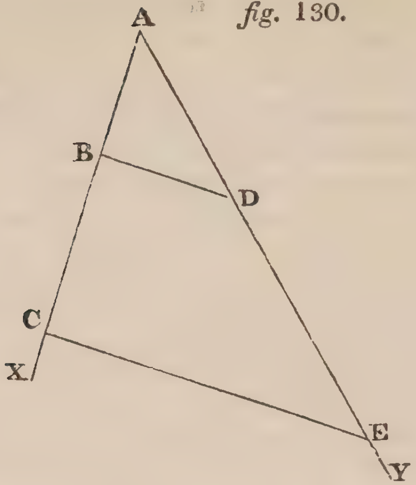
For the rectangle under A B and B C is equal to the rectangle under D B and B E (266.). Hence (262.)

$$DB : AB = BC : BE;$$

that is, $$a : b = c : BE.$$

III. The problem may also be solved thus: — Draw any two lines A X (*fig.* 130.) and A Y forming any angle with each other; take upon A X from A two parts A B and A C, equal to the given extreme a and to one of the means b; on the other line A Y take a part A D from A equal to the other mean c; draw a line joining B and D, and from C draw another line parallel to B D which will meet A Y at E; A E will then be the fourth proportional sought.

fig. 130.



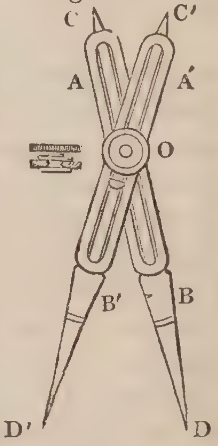
For since B D is parallel to C E, we shall have (248.)

$$AB : AC = AD : AE;$$

that is, $a : b = c : AE.$

(272.) The proportional compasses are an instrument by which the problem for the determination of proportional lines may be always solved.

fig. 131.



This instrument (*fig.* 131.) consists of two similar and equal pieces of brass, A B and A′B′, terminated at each end with steel points, C D and C′ D′. O is a pivot which may be adjusted so as to divide the length of the legs from point to point in any required proportion. In whatever proportion the pivot O divides the legs, in the same proportion will be the distances between the points, to whatever extent the compasses may be opened.

Thus suppose the pivot O is so adjusted that D O shall be twice C O and D′O twice C′O, then the distance D D′ will be twice the distance C C′.

For in the triangles D O D′ and C O C′ the sides including the angles O are proportional to each other, and

the angles O are equal; therefore the triangles are similar, and therefore the sides are proportional. Hence

$$CO : DO = CC' : DD'.$$

If then DO be twice CO, DD′ will be twice CC′; and in general whatever be the ratio of CO to DO, the same will be the ratio of CC′ to DD′.

The legs of the proportional compasses are usually graduated, and the moveable pivot furnished with an index, so that the instrument may be so adjusted as to give any required proportion.

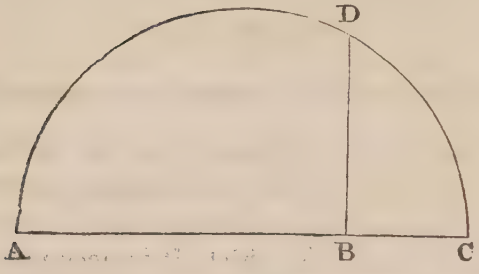
To illustrate the use of this instrument, let us suppose that it is required to draw a line from a certain point which shall be $\frac{3}{10}$ of a given line: let the pivot O be so adjusted that the legs of the compasses be in the proportion of 3 to 10, and let the longer legs be then opened until their points correspond with the extremities of the given line; the distance between the points of the shorter legs will then be $\frac{3}{10}$ of the given line, and this distance may be taken from the given point by means of the compasses.

(273.) Every method by which a fourth proportional to three lines may be found will also be sufficient to find a third proportional to two lines; since the question of a third proportional is reduced to that of a fourth proportional by repeating the mean, and considering it as the case in which the means b and c in the preceding solutions are equal.

(274.) When of three continued proportionals the first and third are given, it is sometimes required to find the second; in other words, it is required to find a mean proportional between two given extremes.

Of the solutions which may be given to this problem the following is the most simple: — Let the given extremes A B (*fig.* 132.) and B C be placed in the same straight line, and on this line A C let a semicircle be described; from the point B draw a perpendicular to A C, meeting the semicircle; B D will then be the mean required.

fig. 132.



It is evident from what has been already proved (267.) that B D is a mean proportional between A B and B C.

(275.) Hence a line may be found whose square is equal to a given rectangle; for it is only necessary to find a line which shall be a mean proportional between the sides of the rectangle (265.).

(276.) The principles which have been established are sufficient for the geometrical solution of the quadrature of any figure formed by right lines; that is, for finding a line whose square shall be equal to the area of such a figure.

It has been shown in (275.) that a line may be found whose square is equal to a given rectangle.

It has been shown in (214.) that a rectangle whose area is equal to that of a given triangle, may be found by constructing one with the same base as the triangle and half its altitude.

It has been shown in (256.) that every right-lined figure may be resolved into triangles: since, then, rectangles may be found whose areas are equal to these triangles severally, and since squares may be found equal to these rectangles severally, and since one square may be found which shall be equal to all these squares taken together (236.), it follows that a square may be found whose area shall be equal to that of the proposed figure.

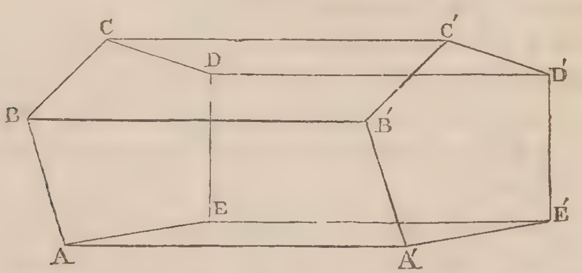
CHAP. XI.

OF THE CONSTRUCTION OF EQUAL AND SIMILAR FIGURES.

THE construction of figures equal or similar one to another, or, in other words, the changing of the position or scale of figures, is of extensive use in the arts; and the various methods by which it is accomplished have an immediate dependence on geometrical principles.

(277.) Let it be required to draw a figure precisely equal and similar, and similarly placed, to the figure A B C D E (*fig.* 133.).

fig. 133.



From the several angles A, B, C, D, E, draw parallel lines to the place where the equal figure is intended to be constructed; and supposing the point A′ to be that at which it is required to place the angle of the figure which corresponds to A; from A′ draw A′ B′ parallel to A B, and meeting the parallel from B at B′; from B′ draw B′C′ parallel to B C, and meeting the parallel from C at C′; from C′ draw C′ D′ parallel to C D, and meeting the parallel from D at D′; from D′ draw D′ E′ parallel to D E, and meeting the parallel from E at E′; lastly, join A′ E′: then the figure A′ B′ C′ D′ E′ will be in all respects equal and similar to the figure A B C D E.

For A B B′ A′ is, by the construction, a parallelogram: therefore A′ B′ is equal to A B; in the same manner B′ C′ may be proved to be equal to B C. The two angles into which the angle A′ B′ C′ is divided by

the continuation of the parallel B B′, are respectively equal to the two angles into which A B C is divided by the parallel B B′; for they are the external angles formed by the parallels and the lines which cross them (41.); therefore the angle A′ B′ C′ will be equal to the angle A B C. In the same manner it may shown that the line C′ D′ is equal to the line C D, and the angle B′C′D′ equal to the angle B C D; also, that the line D′ E′ is equal to the line D E, and that the angle C′ D′ E′ is equal to the angle C D E.

But since A B B′ A′ is a parallelogram, A A′ is equal to B B′; and in the same manner we have the following equalities: —

$$B B' = C C'$$
$$C C' = D D'$$
$$D D' = E E'$$

Hence it follows that

$$A A' = E E'$$

Therefore A A′ E′ E is a parallelogram, and therefore A′ E′ is equal and parallel to A E; and it may be shown, that the angles B′ A′ E′ and A′ E′ D′ are respectively equal to the angles B A E and A E D.

Therefore the figure A′ B′ C′ D′ E′ is in all respects equal and similar to A B C D E.

(278.) By a process analogous to the preceding, a figure may be constructed similar to a given figure, and having its sides in any proposed ratio to those of the given figure.

Let A B C D E (*fig.* 134.) be the given figure, and

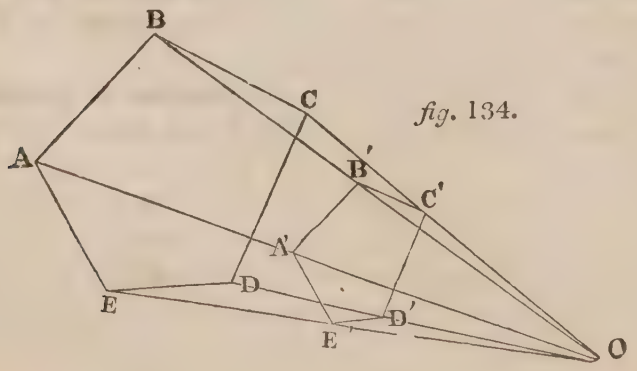


fig. 134.

let it be required to construct another whose sides shall have to the sides of this a given ratio, and shall have the angle corresponding to A at A′.

Draw A A′, and produce it to O, so that the ratio of A O to A′ O shall be that of the sides of the two figures; from O draw O B, O C, O D, and O E; now draw A′B′ parallel to A B, B′ C′ to B C, C′ D′ to C D, and D′ E′ to D E, and join the points A′ and E′. The figure A′ B′ C′ D′ E′ will then be similar to the figure A B C D E, and their corresponding sides will have the required ratio of A′ O to A O.

For since A′ B′ and B′ C′ are parallel to A B and B C, the angle A′ B′ C′ may be proved to be equal to the angle A B C, in the same manner as the corresponding angles were proved to be equal in (277.).

And in the same manner the angles C′ and D′ may be proved to be equal to the angles C and D.

Since A′ B′ is parallel to A B, the triangle A′ O B′ is similar to the triangle A O B; and therefore

$$A'B' : AB = A'O : AO;$$

that is, the corresponding sides A′ B′ and A B of the two figures are in the required ratio. But, for the same reason, we have also

$$A'O : AO = B'O : BO;$$

and since B′ C′ is parallel to B C, we have

$$B'C' : BC = B'O : BO,$$

and therefore

$$B'C' : BC = A'O : AO;$$

that is, the corresponding sides B′ C′ and B C are in the required ratio; and in the same manner it may be shown that the sides C′ D′ and C D, and also D′ E′ and D E are in the required ratio.

But in consequence of the succession of parallels to the sides of the figure A B C D E, we have

$$A'O : AO = B'O : BO$$
$$B'O : BO = C'O : CO$$
$$C'O : CO = D'O : DO$$
$$D'O : DO = E'O : EO.$$

Therefore it follows that

$$A'O : A O = E'O : E O;$$

and therefore (247.) $A'E'$ is parallel to $A E$; and it may be shown, as with the other sides, that the angles A' and E' are respectively equal to the angles A and E, and that the sides $A'E'$ and $A E$ are in the required ratio; therefore the figure $A'B'C'D'E'$ is similar to the figure A B C D E, and has its sides in the required ratio to those of A B C D E.

(279.) When, in the arts, it is required to make a figure equal and similar to a given one, it is frequently done by the process which forms the fundamental test of equality in geometry,—the process of super-position.

Thin transparent paper, called tracing paper, is laid over the figure to be copied; and the sides of the figure being seen through the paper, corresponding marks are made with a pencil on the tracing paper, and the figure is delineated upon it. This process is applicable to all figures, whether bounded by right lines or curves.

The figure thus made on the tracing paper may be transferred again to drawing paper, or to any other surface, by stretching the tracing paper over such surface and marking the outline of the figure by a pencil or pen, penetrating the tracing paper; or a pattern may be cut in card or wood from the figure taken upon the tracing paper, and this pattern being laid upon the surface to which it is required to transfer the figure will give the means of tracing the figure, its sides affording so many rulers by which the chalk, pen, or pencil may be guided.

This process is so common in all the arts, that it is needless to multiply examples of its application; the method pursued by tailors and dress-makers in cutting out clothes, by carpenters, workers in metal and iron, will occur to every mind.

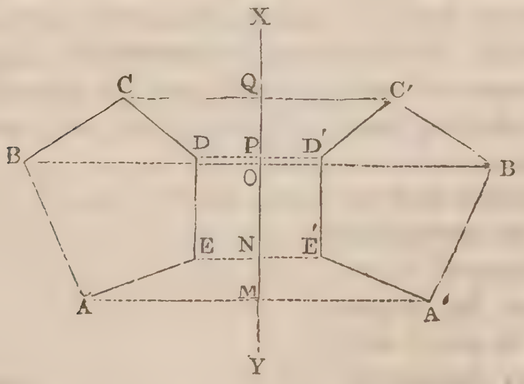
In every species of printing, including letter-press printing, and the printing of engraving in all its forms, the process of super-position is applied; but owing to the surfaces brought together being turned in different

directions, one being presented upwards and the other downwards, the design, when viewed upon them, will be laterally reversed, the points to the right of one corresponding with those to the left of the other. Thus, in letter-press printing, the types which form the words composing a line are put together by the compositor with their faces upwards, but, in impressing the paper, are turned downwards, so that the letters which were on the left when turned upwards will be on the right when turned downwards; but since they leave their impressions on the paper in the order in which they stand when turned downwards, it follows that in order that the printed lines should be read from left to right, the types which produce them must be set from right to left.

The same observation will be applicable to every design printed from types, or from engraving of any kind; and accordingly the plate on which an engraving of a picture or other design is made, must be engraved in a position laterally opposite to that of the picture or design itself.

(280.) A geometrical figure may be laterally reversed by such a geometrical construction as the following:— Let A B C D E (*fig.* 135.) be the figure, and let A′ be

fig. 135.



the point to which it is required that A should be transferred; draw the line A A′, and bisect it. Let M be the point of bisection; through M draw the indefinite

line X Y perpendicular to A A′; from the points B, C, D, E, draw the lines B O, C Q, D P, E N, perpendicular to X Y; and produce each of these lines to the points B′, C′, D′, E′, until the parts of each of them on the right of X Y shall be equal to the parts on the left, that is, so that O B′ shall equal O B, Q C′ shall equal Q C, P D′ equal P D, and N E′ equal N E. Let the points A′, B′, C′, D′, E′ then be connected by straight lines; the figure thus formed will be the figure A B C D E reversed.

For if we conceive the figure A B C D E to be doubled over to the right by a fold along the line X Y, the several perpendiculars from the points M, N, O, P, Q on the left of X Y will fall upon those on the right of it; and since M A is equal to M A′, the point A will fall upon the point A′; and since O B is equal to O B′, the point B will fall upon the point B′; and since Q C is equal to Q C′, the point C will fall upon the point C′; and since P D is equal to P D′, the point D will fall upon the point D′; and since N E is equal to N E′, the point E will fall upon the point E′.

Since then the vertices of each of the angles of the one figure will fall upon the vertices of each of the angles of the other, the one figure when turned over so as to be laterally reversed will exactly cover the other.

If it were required to produce an engraved plate, which, by printing, would give an impression of the figure A′B′C′D′E′, it would therefore be necessary to engrave upon it the figure A B C D E.

In the same manner, if it were required to produce an impression by printing of the figure represented in *fig.* 137., it would be necessary to engrave the plate in the manner represented in *fig.* 136.

fig. 136.

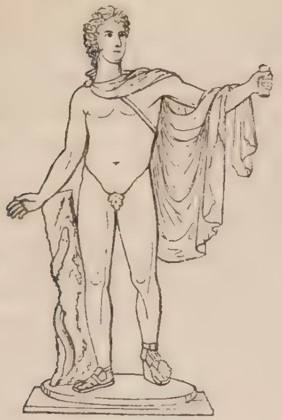
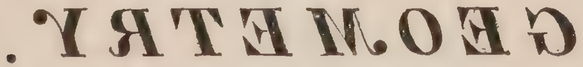


fig. 137.

In the same manner, if it were required to print the word

GEOMETRY,

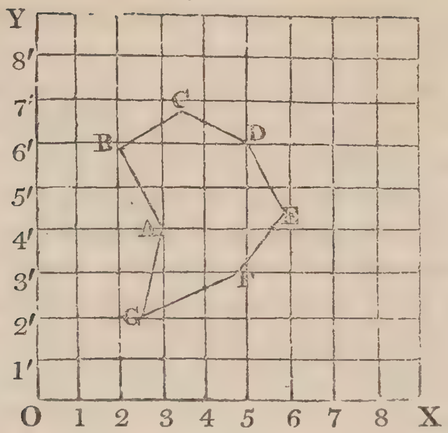
it would be necessary to form and arrange the types thus,



(281.) It frequently happens that it is necessary to copy figures either in their proper position or in a reversed one, under circumstances in which the geometrical methods above explained would be inapplicable: the copy may always be executed by resolving the space occupied by the figure to be copied into a system of squares, by drawing two systems of parallel lines at equal distances, and at right angles to each other, and by drawing a similar system of squares on the surface which is destined to receive the copy. These systems of squares respectively may be removed or obliterated after the copy has been executed.

To render this intelligible, let A B C D E F G (*fig.* 138.) be the figure to be copied: from a point O draw two indefinite lines O X and O Y at right angles, so as to include the figure between them. On O X take O 1, equal to the magnitude of the sides of the squares into which it is

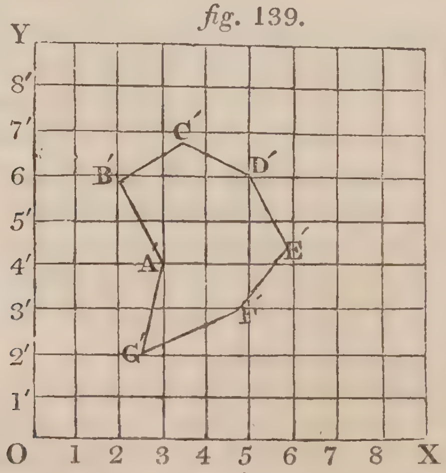
fig. 138.



desired to resolve the space, and repeat this magnitude O 1 along the line O X, so that the line shall be divided into equal parts at the points marked 1, 2, 3, 4, 5, 6, 7, 8; and let the line O Y be similarly divided at the points 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′.

From the points 1, 2, 3, 4, 5, 6, 7, 8, let lines be drawn parallel to O Y; and from the points 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′, let lines be drawn parallel to the line O X. Let similar systems of parallels be drawn upon the surface where it is required to draw a copy of the figure, as in *fig.* 139.

fig. 139.



Commencing with the point A, we find that it is at the intersection of the fourth parallel from O X, and the

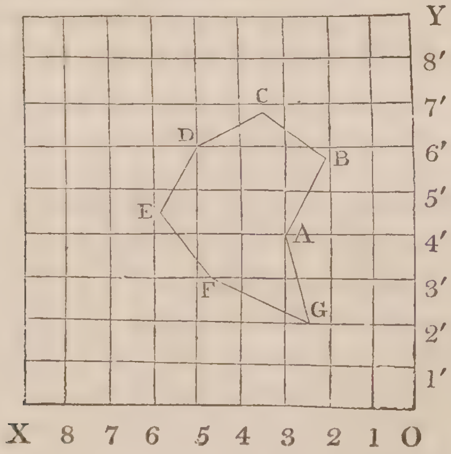
third parallel from O Y: its position on the surface on which the copy is to be made is determined, and is at A′ (*fig.* 139.).

We find the point B on the second parallel from O Y at a certain distance above the fifth parallel from O X; let that distance be taken in the compasses, and a similar distance be taken on the second parallel from O Y, in *fig.* 139., above the fifth parallel from O X.

Again, we find the point C to the right of the third parallel from O Y, and above the sixth parallel from O X. Taking in the compasses its distances from these two parallels, the position of the corresponding point in *fig.* 139. will be determined; and in the same manner the position of each angle or other point may be found; and by connecting the several points thus determined an equal and similar figure will be formed.

(282.) If it be required to construct a figure laterally reversed, the system of squares must likewise be reversed, and the points of the required reversed copy may be determined in the same manner (*fig.* 140.).

fig. 140.



By a slight modification of this process, copies may be made on an increased or diminished scale in any required proportion; it is only necessary that the squares used with the figures respectively shall have their sides in the required proportion, and that the distances of the

points between the parallels shall be taken in the same proportion. Let us suppose that a painting or drawing on a large scale is required to be copied on a small scale: a frame consisting of threads of silk or wire stretched parallel to each other at equal distances, and at right angles, is placed before the picture to be copied; let a frame consisting of an equal number of squares, whose sides shall be less in the required proportion, be placed over the paper or canvass on which the reduced copy is intended to be made; the proportion of the points of the drawing corresponding to those of the original will be determined according to the position of the square in which they are found, and by their position in that square.

The proportional compasses (*fig.* 131.) furnish an easy and accurate means of determining the position of the points in their respective squares. Let the legs of this instrument be so adjusted by the moveable centre, that the ratio of their lengths shall correspond to the proportion in which the picture is to be reduced; and when the distance of any point of the original from an adjacent parallel is taken by the longer legs, the opening of the shorter legs will give the distance of the corresponding point of the copy from its adjacent parallel.

(283.) In ornamental needle-work, the same system of copying is practised: the figures to be executed are usually required to be wrought on coarse canvass, the threads of which form a system of squares, such as have been just described. The original object from which the copy is made is delineated in proper colours on paper on which a similar system of squares is printed, the colour occupying each square being there distinctly expressed. The square printed on the paper corresponding with the squares formed by the threads of the canvass, the colour occupying each square on the paper directs the needle-worker in the choice of the colour of the silk or worsted with which the corresponding square on the canvass should be filled.

It is evident that the state of the work in this case

will be regulated by the coarseness or fineness of the canvass; the coarser canvass giving figures on a larger scale.

If the figures are required to be wrought upon a cloth, of which the threads do not form the necessary system of squares, the work may be executed by stretching over the cloth on which the figures are to be wrought, canvass of that degree of coarseness which will give the required degree of magnitude to the pattern. The needle is then passed through both the canvass and the cloth on which the pattern is intended to be worked, so that in the first instance the effect of the work is to stitch together the canvass and the cloth. The threads of the canvass are, however, subsequently drawn out one by one, and the pattern is exhibited wrought in its proper form and colours upon the cloth.

The reduction and the reversing of designs is much used in the art of engraving. When it is required to produce an engraving from a picture upon a large scale, a copy of the painting must, in the first instance, be made on the scale on which it is intended to be engraved. This copy should represent the original in its true proportions. By habit the eye of an engraver acquires extraordinary skill and quickness in the detection even of very slight deviations from the just proportions in such reduced copies. We have known a case in which an artist, who executed a celebrated picture, on examining the copy which had been made for the engraver, was satisfied with its accuracy, yet the moment the copy was submitted to the inspection of the engraver, small inaccuracies of proportion were apparent to his eye which could not be discerned by the original artist, but which were rendered evident by the application of the proportional compasses. Such copies should therefore always be tested by this instrument.

(284.) Among the properties of geometrical figures some of the most striking, by their beauty and generality, and by their application in the arts, do not admit of demonstration by any principles of mathematical reasoning

sufficiently simple and elementary to be introduced with propriety into this volume. As examples of such properties, the following may be mentioned: —

A regular polygon contains a greater area than any other figure of the same perimeter and the same number of sides.

If two regular polygons have equal perimeters, that which has the greater number of sides contains the greater area.

Of all figures having the same perimeter, whether the sides be straight or curved, that which contains the greatest area is the circle.

The following examples will illustrate the application of these properties in the useful arts: —

When a Gothic window of a given magnitude consists of panes of glass having the form of a regular polygon, a less amount of metallic framing will be required than if the frames had any other figure; and the more numerous the sides of the polygonal frame, the less will be the quantity of framing. As the only regular figures by which a space can be covered are the triangle, square, and hexagon (195.), it follows that the form of pane requiring the least quantity of framing is the hexagon.

In the construction of pipes for the conveyance of gas, water, or other fluid, the object is to convey the greatest quantity of the fluid with the least expense of pipes. Although there are other reasons which render the use of circular pipes expedient, they have the further advantage of containing, in a given length and with a given quantity of material, a greater quantity of fluid than pipes of any other form.

CHAP. XII.

OF STRAIGHT LINES AND PLANES.

(285.) The relations and properties of geometrical figures which have been explained in the preceding chapters are those which belong to straight lines and circles, supposed to be described on the same plane surface. Thus, when several straight lines have been considered, or when straight lines have been viewed in relation to circles, or circles regarded in relation one to another, each line or circle has been understood to be so placed, that the plane surface on which it is formed is the same as that on which the other lines or circles under inquiry are also formed.

It is, however, frequently necessary to investigate the geometrical relations of lines which do not lie in the same plane.

It has been stated (5.), that one of the properties of a plane surface is, that any two points in it being united by a straight line, every part of that straight line, whether between the two points or beyond them, however far the line be continued, will lie in the plane surface. From this property another immediately flows. If two plane surfaces intersect each other, their line of intersection will be *straight*. For, let a straight line be imagined to be drawn from any one common point of the two planes to any other; every part of that straight line must, in virtue of the property just referred to, lie at the same time in both planes, and must, therefore, be their common line of intersection.

That the intersection of two plane surfaces is a straight line, is a proposition which, when applied to any particular case, becomes so evident, that it can hardly be considered to require proof. The walls of a

room being plane surfaces, the corners formed by their intersection are straight lines. In like manner, the lines formed by the junction of the floor with the walls are straight. The surfaces which form the sides of an obelisk are plane, and its corners are straight lines.

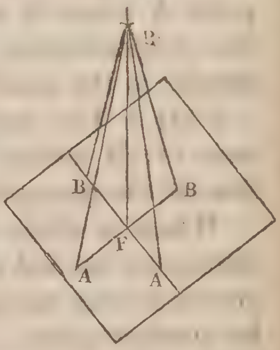
On the other hand, if a plane surface intersect a curved surface, the line of intersection will be *generally* curved. Thus the surface forming the side of a bridge intersecting the curved surface forming the arch of the bridge, the line of intersection forming the corners of the arch is curved, and the species of curve depends on the form of the arch.

The line of intersection of a plane and curved surface, however, *may* be straight. Thus, if a circular pillar be cut by a plane along its centre, the lines of intersection of the curved surface of the pillar, and the plane surface formed by the section, will be straight.

Thus, the intersection of curved surfaces *may* be a straight line; but the intersection of plane surfaces *must* be a straight line.

(286.) If a point P (*fig.* 141.) be assumed any where above a plane, there will be a certain point F upon the plane which is nearer to P than any other point on the plane. The line P F will in that case be perpendicular to every line, such as A F B, drawn through the point F upon the plane.

fig. 141.



For since P F is the shortest line which can be drawn from P to the plane, and since every point of the line A B is in the plane, P F must be the shortest line which can be drawn from the point P to the line A B, and must therefore be perpendicular to the line A B.

(287.) A line, such as P F, which is thus perpendicular to every line that can be drawn in a plane through the point where it meets the plane, is said to be perpendicular to the plane itself.

The point F, where the perpendicular meets the plane, is called the *foot of the perpendicular.*

(288.) All lines, such as P A, P B, drawn to a plane from a point P above it, which are equally inclined to the perpendicular to the plane, are equal.

For in the triangle P F A and P F B, the side P F is common, the angles at F, being right, are equal, and the angles at P are supposed equal. Therefore by (61.) the triangles must be in all respects equal, and therefore P A must be equal to P B.

(289.) It follows, also, that the points A, B, where such lines meet the plane, are equidistant from F, the foot of the perpendicular.

In fact, if a straight line P A revolve round the perpendicular P F, always making with it the same angle, the part of that straight line between the point P and the plane will continue of the same length, and it will, as it revolves, describe a circle, on the plane of which F will be the centre.

(290.) The greater the angle is which the line P A makes with the perpendicular, the greater the line P A will be, and the greater also will be the distance of the point A where it meets the plane from F, the foot of the perpendicular. This may be shown in a manner similar to the proof of the analogous property in (24.).

(291.) The perpendicular to a plane is called the *axis* of all circles described on that plane, round the foot of the perpendicular as a centre.

When a circle revolves round its axis, the figure undergoes no real change of position, each point of the circumference taking successively the position deserted by another point.

On this geometrical principle is founded the operation of millstones. Two circular stones are placed so as to have the same axis, to which their faces are perpendicular, being therefore parallel to each other and regulated in their distance according to the fineness of the flour intended to be ground. The inferior stone is fixed, while the superior stone is made to revolve by

the power which drives the mill. The relative position of the circular faces of the millstones undergoes no real change during the revolution, and their distance being properly regulated, all the corn which passes between them will be ground with the same fineness.

The advantage, and even the necessity of great precision in the construction of machinery is strikingly illustrated by the effects of any want of exactitude in the position of millstones. If the parallelism of the faces of the stones be not perfect, — if the axis of the moving stone be not truly at right angles to its circular face, — the two grinding surfaces will not be at one uniform distance, and the relative position of the two stones, instead of being uniform, will constantly vary. The grain will be, therefore, differently affected by them, one part not being ground at all, or not sufficiently so, and another part being too much broken, and perhaps heated and spoiled.

In the lathe, the axis round which the body to be turned is made to revolve, is the axis of the circles, which the cutting instrument forms by removing the matter which projects beyond the proper distance from that axis. The process of turning, therefore, consists in the formation of a surface, the cross section of every part of which is a circle, all the circles having the same axis.

(292.) Since two perpendiculars to the same plane are both perpendicular to the same straight line, in that plane joining their feet, they must be parallel to each other (28.), and hence all perpendiculars to the same plane are parallel to each other.

(293.) The plane which the surface of a liquid in a state of quiescence forms is an *horizontal plane,* and if indefinitely continued in all directions around, is called the *plane of the horizon.*

If a weight be suspended by a flexible string so as to form a *plumb line,* such string, when the weight is at rest, will have a direction perpendicular to a horizontal plane. The line of direction of such a string is called a *vertical line.*

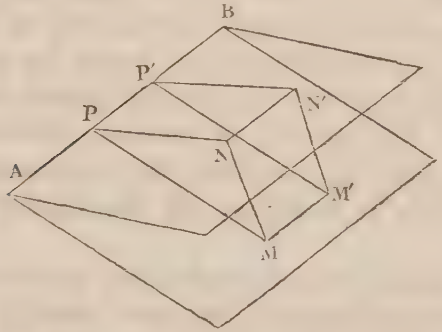
Since all vertical lines are perpendicular to the same horizontal plane, they will be parallel to each other, provided the distance between them be not so great as to cause a sensible effect to be produced by the curvature of the earth's surface.

(294.) When two planes intersect they may be more or less inclined one to the other. The angle which they form with one another, is the angle formed by two straight lines drawn from any one point in their line of intersection perpendicular to that line, one being drawn on the one plane, and the other on the other.

Thus, if the straight line A B (*fig.* 142.) be the line formed by the intersection of the two planes, take P any point on that line, and draw P M in the one plane and P N in the other, both perpendicular to A B. The angle M P N is the angle formed by the planes.

It is easy to shew that wherever the point P be taken

fig. 142.



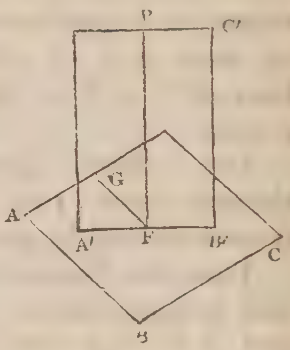
upon the line A B, the angle M P N will be the same. From any other point P′, let P′ M′ and P′ N′ be drawn also perpendicular to A B. The angle M′ P′ N′ will be equal to M P N. For take P M equal to P′ M′, and P N equal to P′ N′, and draw M M′, N N′, M N, M′ N′. Since P M and P′ M′ are equal and parallel, M M′ and P P′ will be equal and parallel, and for a similar reason N N′ and P P′ are equal and parallel, and therefore N N′ and M M′ are equal and parallel, and therefore M N and M′ N′ are equal and parallel. The triangle M P N has therefore its three sides respectively equal to those

of the triangle M′ P′ N′; and therefore (62.) the angle M P N will be equal to the angle M′ P′ N′.

(295.) If a straight line be perpendicular to a plane, any plane drawn through that straight line will be also perpendicular to the plane.

Let the line P F (*fig.* 143.) be perpendicular to the plane A B C, and let the plane A′ B′ C′ be drawn through the line P F, then the plane A′ B′ C′ will be perpendicular to the plane A B C; for let F G be drawn in the plane A B C perpendicular to A′ B′, then, since P F is perpendicular to the plane A B C, the angle G F P will be a right angle (286.), and the lines G F and P F, being both perpendicular to A′ B′, the right angle G F P is the inclination of the two planes (294.), and the planes are therefore perpendicular.

fig. 143.



If the plane A′ B′ C′ be supposed to turn round on the line F P as an axis, it will be perpendicular to the plane A B C in all its positions.

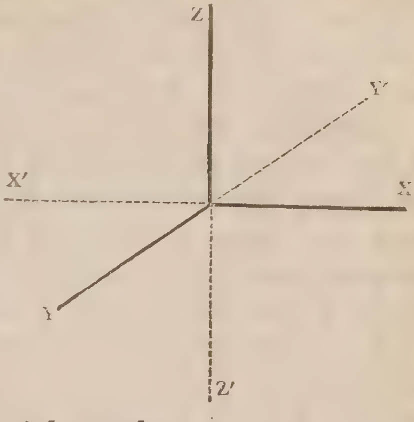
Thus, a door turning on its hinges is perpendicular to the floor and ceiling of the room in every position which it can assume. As it turns, it changes its inclination to the wall in which it is constructed, the angle of inclination being that which is formed by the edge of the top of the door, and of the corresponding edge of the top of the door-frame.

(296.) If a vertical line be drawn from any point in an horizontal plane, all planes passing through that vertical line will be perpendicular to the horizontal planes. Such planes are called *vertical planes*.

(297.) If two lines be drawn on a plane at right angles to each other, and a third line be drawn from the point where these two lines cross each other perpendicular to the plane, then these three right lines will be perpendicular to each other.

If the straight lines X X′ and Y Y′ (*fig.* 144.) be supposed to be drawn at right angles on a horizontal plane, and Z Z′ be drawn vertically, or perpendicular to that plane, through the point O, where X X′ and Y Y′ intersect, the angles formed by each pair of these lines at O will be right angles.

fig. 144.



The three planes through each pair of these lines will be also at right angles.

Thus the horizontal plane through X X′ and Y Y′ will be perpendicular to the vertical plane through Z Z′ and X X′, and also to the vertical plane through Z Z′ and Y Y′.

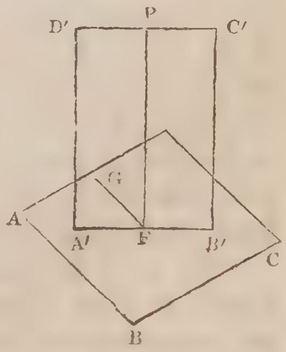
The two vertical planes through Z Z′, X X′, and through Z Z′, Y Y′, will also be at right angles.

The right lines which intersect at O form round that point twelve right angles, four being formed on each of the three rectangular planes.

Three rectangular planes, and no more, can therefore be always drawn through the same point.

(298.) If two points, such as C′ D′ (*fig.* 145.), be taken at equal distances from a plane A B C, every point of the straight line drawn through these points C′ D′ will be at the same distance from that plane.

fig. 145.



For if from C′ D′, perpendiculars C′ B′ and D′ A′ be drawn to the plane, and the line A′ B′ be drawn; then A′ B′ C′ D′ will be a parallelogram, and C′ D′ will

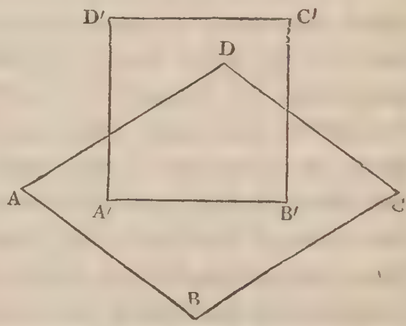
be parallel to B′ A′. All perpendiculars drawn from C′ D′, or from its continuation in either direction, will be equal to the perpendiculars C′ B′ and D′ A′; and they will be perpendicular to the plane A B C, and will therefore be the distances of the points in C′ D′, or its continuation from the plane A B C.

(299.) A line is said to be parallel to a plane when all its points are thus equally distant from the plane.

(300.) If a plane A′ B′ C′ D′ (*fig.* 146.) pass through a line, such as C′ D′, parallel to another plane A B C D, then the intersection A′ B′ of these two planes will be parallel to C′ D′, whatever be the angle the two planes make with each other.

For, since C′ D′ is parallel to the plane A B C D, it can never meet that plane, however it may be prolonged; and therefore cannot meet any line drawn in that plane. It cannot, therefore, meet the line A′ B′, formed by the intersection of the two planes. The two lines A′ B′ and C′ D′ can never, therefore, meet; and since they are both in the same plane, they must be parallel.

fig. 146.



(301.) It may be here observed, that the conditions under which two straight lines are parallel are twofold: first, they must be both in the same plane; and, secondly, their directions must be such, that, however they may be prolonged in either direction, they can never meet. It is easy to conceive two lines differing very much in direction, and therefore not parallel, but which nevertheless can never meet, however they may be prolonged: thus, if from any point in a horizontal plane a vertical line be drawn, and from another point in the same plane, lying north of the former point, a line be drawn east and west; these two lines will evidently not be

parallel, and yet however prolonged, they can never meet.

(302.) Three points, however they may be placed, must always lie in the same plane. For if a straight line be drawn, uniting two of them, and a plane be drawn through that line, and be made to revolve upon it as an axis, it must, at some point of its revolution, pass through the third point; in that position therefore of the plane, the third point will be in it.

(303.) If more than three points be considered, they may or may not be in the same plane, since the fourth may be above or below the plane through the other three.

(304.) It is on this geometrical principle that stability in practice is more readily obtained by three supports than by a greater number. A three-legged stool must be steady if placed on a plane surface, since the ends of its legs, being in the same plane, will always accommodate themselves to the surface which supports it; but if the stool have four legs, the end of one of these may not be in the same plane with the ends of the other three, in which case it will be unstable, since the ends of the four legs cannot possibly at the same time rest on the surface which supports the stool. In well constructed furniture, the ends of the legs are formed in the same plane, and therefore four or more legs are used; but in rudely constructed stools and tables it is not unusual to form them with three legs, the inequality of length being then not a cause of instability.

(305.) The use of three rectangular planes, such as those described in (297.), is very frequent in the arts, and especially in architecture, carpentering, and the other departments of art relating to buildings. The floor and walls of a room present an obvious example of a system of such planes; beams of wood, bricks, blocks of stone, and almost all the materials used in building, afford like examples.

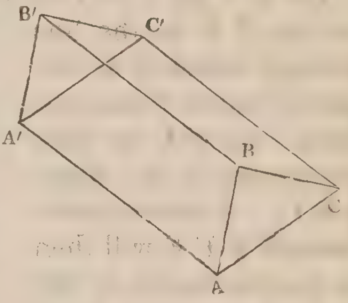
In architectural and mechanical drawing, it is usual to represent buildings and machines by views taken of

them in the direction of three rectangular planes: the view taken in the horizontal plane, is called the ground plan, in addition to which a view is taken in two vertical planes at right angles to each other: if these views exhibit the exterior of the object, they are called *elevations;* if they show its interior, they are called *sections.*

(306.) If three points be taken at equal distances above a plane, the plane which passes through these three points will be parallel to the former plane.

Let the three points be A′, B′, C′, taken at equal distances above the plane A B C (*fig.* 147.); and from them let three perpendiculars A A′, B B′, and C C′, be drawn to the plane, these three perpendiculars will be equal and parallel, and therefore A B will be parallel to A′ B′, B C will be parallel to B′ C′, and A C will be parallel to A′ C′. These three lines, however prolonged in the one plane, can never therefore meet the other plane, and therefore the planes themselves can never meet; for if they did, one or other of the three lines joining the three given points must meet the line of intersection of the planes, since all the three lines could not be parallel to that line, and therefore one of them would meet the other plane, contrary to what has been proved.

fig. 147.



(307.) If two planes be parallel one to the other, they will be every where equally distant from one another.

For if any two points in the one be at unequal distances from the other, let perpendiculars be drawn from these points to the other plane. The line joining the tops of these perpendiculars in the one plane, will therefore not be parallel to the line joining their feet in the other plane; these two lines would therefore meet if continued, and therefore the planes in which they are drawn would meet, and could not be parallel; all points,

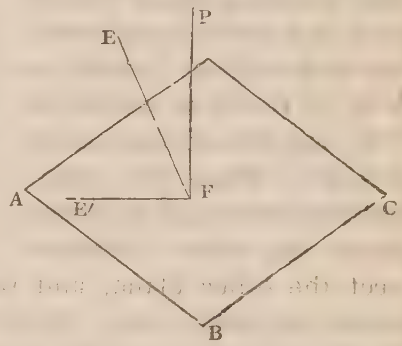
therefore, in a plane parallel to another, will be equally distant from that other.

(308.) If two parallel planes be both intersected by a third plane, their lines of intersection with that third plane will be parallel. For since the parallel planes on which these lines are drawn do not meet, the lines themselves can never meet, and since they are both in the third, or intersecting plane, they must be parallel.

(309.) If from a point in a plane, any straight line be drawn, not lying in that plane, another plane may be drawn passing through that line, which shall be perpendicular to the former plane.

Let F (*fig.* 148.) be the point in the plane A B C, and let F E be the line through which it is required to draw the second plane; let F P be perpendicular to the plane A B C. A plane drawn through P F E will then be perpendicular to the plane A B C (295.). If the line F E, through which it is required to draw a plane perpendicular to the plane A B C, be itself perpendicular to the plane A B C, it will then be identical with F P, and any plane whatever drawn through it will be perpendicular to A B C; but if it form any angle with F P, then only one such plane can be drawn.

fig. 148.



If the line F E be perpendicular to F P, it will then be in the given plane A B C, but the solution of the question will be the same.

(310.) The angle which a line such as F E makes with a plane A B C, which it meets at F, is the angle formed by the line F E, and the line F E′ formed by the intersection of the plane through F E perpendicular to the plane A B C.

(311.) The angle under a straight line and a plane

is, therefore, the complement of the angle under that line, and a perpendicular to the plane.

(312.) If two planes be parallel, all lines drawn from the one to the other equally inclined to them are equal.

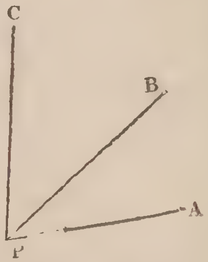
For all such lines will be equally inclined to the perpendicular to these planes, and will therefore be equal.

(313.) Parallel lines intercepted between parallel planes are equal.

For they must be equally inclined to the parallel planes.

(314.) If from a point P (*fig.* 149.) two straight lines PA and PB be drawn, forming any angle A P B, and from the same point a third line PC be drawn, lying above the plane of the angle A P B, this third line will form angles with PB and PA, whose planes will be different from each other, and from the plane of the angle A P B. In fact, the three angles of which the point P is the common vertex, will have their planes mutually inclined to each other. The intersections of these planes, one with another, being the lines P A, P B, and P C, which form the sides of the three angles.

fig. 149.



(315.) The figure thus formed, with its vertex at P, is called a *solid angle.*

The lines PA, PB, and PC, are called the *edges* of the angle.

The plane angles APB, APC, and BPC, of which the solid angle is formed, are called the *faces* of the solid angle.

(316.) Any two angles, A P C and B P C, forming the faces of a solid angle, must be greater together than the third A P B, for if they were not, the line P C could not lie above the plane of the angle A P B.

(317.) It is evident that three rectangular planes

will form eight solid angles round the point O (*fig.* 144.), each of these solid angles having three rectangular faces.

(318.) A solid angle cannot have less than three faces, but it may have four or more.

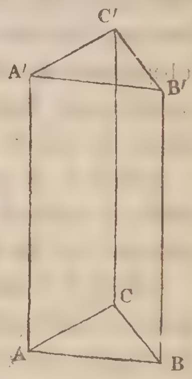
The corners of a room or of a chest are solid angles, with three rectangular faces; the point of a triangular file or of a small sword are solid angles, with three acute faces. The ornamental cutting of glass, and the forms given to precious stones when cut and polished, the forms assumed by natural crystals, all afford examples of solid angles, consisting of various numbers of faces of various magnitudes.

CHAP. XIII.

OF PRISMS AND PYRAMIDS.

(319.) If from three points, A, B, C (*fig.* 150.), taken upon a plane and forming the vertices of a triangle, three equal perpendiculars, A A′, B B′, C C′, be drawn, the points A′, B′, C′, will lie in a plane parallel to the plane of the points A, B, C; and the triangle A′ B′ C′ will be in all respects equal and similar to the triangle A B C.

fig. 150.



A solid figure will thus be formed having equal and similar triangular ends or bases, and three rectangular sides. The edges of its sides A A′, B B′, C C′ are equal and parallel; and the three edges A B, B C, C A of the one end are respectively equal and parallel to the edges A′ B′, B′ C′, C′ A′ of the other end.

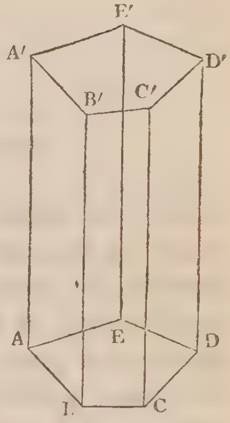
Such a solid is called a *triangular prism.*

(320.) If, instead of drawing three equal perpendiculars from the points A, B, C, three parallel lines not perpendicular to the plane had been drawn, and three points, A′, B′, C′, on these parallels had been taken at equal distances from A, B, C, and were joined so as to form another triangle A′ B′ C′, a solid figure would likewise be formed having equal and similar triangular ends: the sides would in this case be oblique-angled parallelograms. In the former case the planes of the sides would be perpendicular to the planes of the ends; in the present case they will form oblique angles with those planes.

(321.) If four or more points, A, B, C, D, E

(*fig.* 151.), be taken upon the same plane, and from these points parallel lines be drawn not in the plane, and equal distances upon these parallels be taken, so as to determine a similar system of points, A′, B′, C′, D′, E′, in a parallel plane; an equal and similar figure will be formed by joining these points. The solid figure thus constructed having equal and similar ends, and having for its sides as many parallelograms as there are sides to the rectilinear figures which form its ends, is called a *prism*.

fig. 151.



(322.) A prism whose sides are perpendicular to its ends is called a *right prism;* and one whose sides are oblique to its ends is called an *oblique prism.*

(323.) Prisms are denominated *triangular, quadrangular, pentagonal,* &c. &c., according as their ends are *triangles, quadrilaterals, pentagons,* &c.

(324.) A quadrangular prism whose ends are parallelograms (*fig.* 152.) is called a *parallelopiped.*

fig. 152.

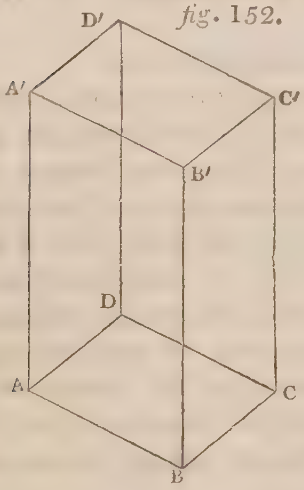
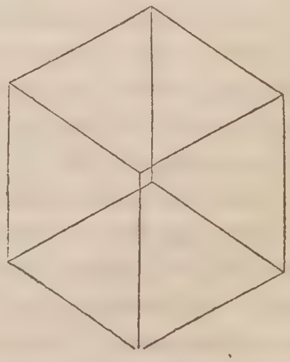


fig. 153.



(325.) A rectangular parallelopiped (*fig.* 153.), whose

ends are squares, and whose height is equal to the side of its end, is called a *cube*.

Dice used in games of chance have the form of cubes.

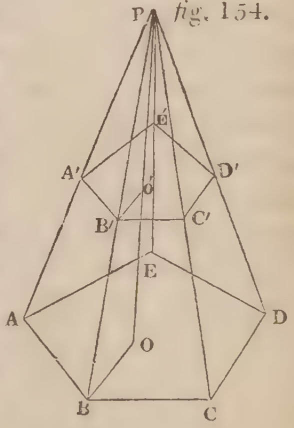
(326.) Every prism may be resolved into as many triangular prisms as the figures forming its ends or bases can be resolved into triangles.

If from any angle of one of the bases diagonals be drawn so as to resolve the base into triangles, and from the corresponding angle of the other base similar diagonals be drawn; the several diagonals of the one base will be parallel and equal to the diagonals of the other base. If planes be drawn through every pair of corresponding diagonals, these planes will resolve the prism into as many triangular prisms as there are triangles in its bases.

(327.) The rectangular parallelopiped is the form of prism most frequently presented in the arts. In masonry, it is the form given to bricks and to hewn stone; in carpentry, it is the form given to beams of timber; in buildings, an oblong rectangular parallelopiped is the most common form for rooms; and since the walls of a building, whatever its plan may be, must be perpendicular to its base, the form of the building must always be that of a right prism.

(328.) If any rectilinear figure, A B C D E (*fig.* 154.) be traced upon a plane, and from any point P above that plane straight lines be drawn to the several angles of the figure; a solid will be formed having the figure A B C D E traced upon the plane for its base, and having as many triangular faces as the base has sides, these triangular faces terminating in the common vertex P, which forms the summit of the figure. Such a solid is called a *pyramid*, the point P being called its *vertex*.

fig. 154.



(329.) Pyramids are denominated *triangular*, *quadrangular*, &c. &c., according to the figures which form their bases.

(330.) Obelisks are pyramids having square bases, and equal and similar triangular sides, the heights being very great in proportion to the magnitudes of their bases.

The Pyramids of Egypt are pyramids having square bases, and similar and equal triangular sides.

(331.) A *regular pyramid* is one which has a regular figure for its base, and its vertex perpendicularly over the centre of the circle which circumscribes its base; thus, a regular triangular prism has an equilateral triangle for its base, and a line drawn from its vertex to the centre of its base will be perpendicular to its base.

(332.) As all plane rectilinear figures admit of having their areas resolved into as many triangles as they have sides, by taking any point within them as the common vertex of the component triangles; so all solids whatever admit of having their volumes resolved into as many pyramids as they have faces, by taking within their volumes any point as the common vertex of the component pyramids, and drawing lines from that point to their several angles, which lines will form the edges of the triangular faces of the component pyramids.

The species of pyramids into which the solid is thus resolved will depend on the kind of figures formed by the faces of the solid figure; but since all pyramids whatever can be resolved into triangular pyramids by drawing planes through their vertices and the diagonals of their bases, it follows that all solids whatever having plane faces bounded by straight edges admit of being ultimately resolved into triangular pyramids.

CHAP. XIV.

OF THE VOLUMES OF SOLID FIGURES.

(333.) The perpendicular drawn between the planes of the bases of a prism is called the *altitude* of the prism.

(334.) If two prisms have equal bases and equal altitudes, they will have equal volumes, whatever may be the form of their bases, or whatever may be the inclination of their bases to their sides.

For the volume of the prism may be considered to be composed of a number of plates indefinitely thin piled one upon the other. The number of plates composing prisms of equal altitudes will be evidently the same, provided the component plates of each have the same thickness. Prisms of equal altitudes being therefore composed of the same number of plates, their volumes will be the same when the component plates have the same superficial magnitude.

This form of demonstration, which is in the spirit of the higher geometry, may be more clearly comprehended by the following illustration: — A pack of cards placed in a perpendicular heap forms a rectangular prism, as represented in *fig.* 155. If they be piled so as to lean towards the end of the pack, as in *fig.* 156., they will still form a prism, having the same base and the same altitude as before. In this case, two of the sides of the prism will be per-

fig. 155.

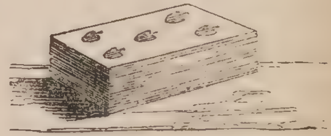
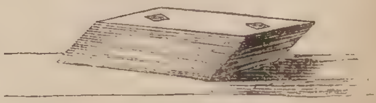


fig. 156.

fig. 157.



pendicular to the base, the other two being oblique to it. In *fig.* 157., the same cards are represented in such a

position as to form a prism in which all the sides are oblique to the base.

(335.) The volume of a prism depends, therefore, conjointly on its altitude and the area of its base. With the same magnitude of base, the volume will increase or diminish in the same proportion as the altitude is increased or diminished; and with the same altitude, the volume will increase or diminish in the same proportion as the base is increased or diminished.

(336.) A pyramid, whatever be the form of its base, may be conceived to be formed of a number of thin plates of matter piled one upon another, gradually diminishing in magnitude upwards until they are reduced to a point at the vertex of the pyramid. The plates thus composing a pyramid will have figures similar to each other and to the base of the pyramid. Thus, a triangular pyramid will be a pile of similar triangles gradually diminishing in magnitude upwards. That this is the case will be made evident by showing that any section of a pyramid made by a plane parallel to its base will be a figure similar to its base. Let the pyramid, *fig.* 154., be cut by a plane passing through the point A′ parallel to its base, and let the section made by this plane and the side of the pyramid be A′B′C′D′E′; since A′B′ is parallel to A B, the ratio of A′ B′ to A B will be that of P B′ to P B, and for the same reason the same will be the ratio of B′C′ to B C. Thus each of the sides of A′B′C′ D′E′ will bear the same ratio to the corresponding side of A B C D E; and the corresponding angles of the figures being also equal each to each, the figures will be similar. This section, A′ B′ C′ D′ E′, may be considered as the surface of one of the plates of which the prism is composed.

From what has just been proved, it is evident that the area of any section of a pyramid parallel to the base will have to the area of the base, the same ratio as the square of its distance from the vertex to the square of the distance of the base from the vertex, these distances being measured along the edges of the pyramid.

For these distances being proportional to the corresponding sides of the similar figures, their squares will be proportional to the squares of those sides; but the area being as the squares of the sides, it follows that they will be as the squares of their distances from the vertex.

If a perpendicular P O be drawn from the vertex to the base of a pyramid, it will be divided at O′ by a plane parallel to the base, in the same proportion as that plane divides other lines drawn from the vertex to the base. For, let B O and B′ O′ be the intersections of the plane of the angle B P O with the plane of the base and the plane of the parallel section; the lines B O and B O′ will then be parallel, and therefore P B will be divided at B′ proportionally to P O at O′.

It follows, therefore, that the area of the section of a pyramid made by a plane parallel to the base, will be in the proportion of the square of the distance of that plane from the vertex.

(337.) If two pyramids have equal bases and equal altitudes, sections of them made by planes parallel to their bases will be equal, if they are at equal distances from their vertices.

For the areas of these sections will have to the areas of their bases, the same ratio as the squares of their distances from the vertices to the squares of their altitudes: these ratios being equal, and their bases being equal, the sections will be equal.

(338.) Two pyramids having equal bases and equal altitudes will have equal volumes; for since they have equal altitudes, they will be composed of the same number of plates; and since the bases are equal, the plates, which are equally distant from the vertices, will be equal. The component plates, therefore, being equal in number, and equal each to each in magnitude, the volumes of the pyramids composed of them will be equal.

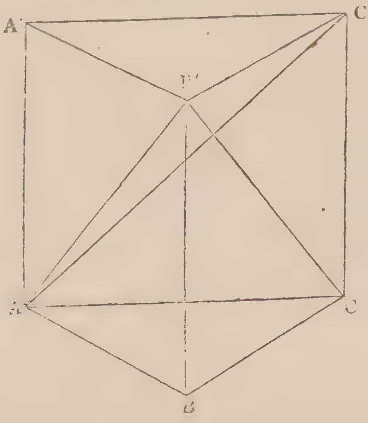
(339.) The volume of a pyramid depends, therefore, conjointly on the magnitude of its base and its altitude. If its altitude remain the same, its volume will increase

or diminish in the same proportion as its base is increased or diminished; for, in that case, it will consist of the same number of plates, all the plates being increased or diminished in the same proportion as its base is increased or diminished. If it have the same base, its volume will increase or diminish in the same proportion as its altitude is increased or diminished; for, in that case, while the magnitude of the corresponding plates remains unaltered, their numbers will be increased or diminished in the same proportion as the altitude is increased or diminished.

(340.) The volume of a triangular prism is equal to three times the volume of a pyramid, which has the same base and altitude as the prism.

Let A B C and A′ B′ C′ (*fig.* 158.) be the two bases

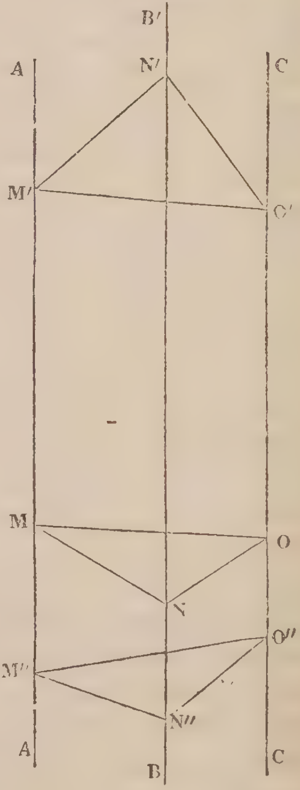
fig. 158.



or ends of the prism, and let a plane be supposed to be drawn through the edge A C and the angle B′; a pyramid will thus be cut off from the prism whose base is A B C, and whose vertex is at B′. If another plane be drawn through the edge B′ C′ and the angle A, a second pyramid will be cut from the prism, having for its base A′ B′ C′, and for its vertex A. The altitude of each of these two pyramids will be the same, being the distance between the bases of the prism; and their bases will be equal, being the ends of the prism. The

remainder of the prism after removing these two pyramids will be the pyramid whose base is A C C′, and whose vertex is B′; but the volume of this pyramid will be equal to the volume of the pyramid whose base is A A′ C′, and whose vertex is B′, because these two pyramids have the equal triangles into which the parallelogram A A′ C′C is divided by its diagonal A C′ for their bases, and have a common vertex B′. It follows, therefore, that the three pyramids into which the prism is divided by the planes A B′ C′ and A B′ C have equal volumes; and since one of these has the base of the prism for its base, and the altitude of the prism for its altitude, the volume of the prism must be equal to three times the volume of the pyramid having the same base and altitude.

fig. 159.



(341.) The volume of any prism whatever is equal to three times the volume of a pyramid having the same altitude, and having a base of equal area; for, whatever be the form of the base of the prism, its volume will be equal to that of a triangular prism having an equal base and altitude.

(342.) A figure formed by the section of a prism by a plane not parallel to its base is called a *truncated prism*.

Let A A′, B B′, C C′ (*fig.* 159.), be the three parallel edges of a triangular prism, and let M N O be the section of that prism by any plane whatever; and let M′ N′ O′ be its section by another plane not parallel to the former. The figure whose ends or bases are M N O and M′ N′ O′ is a *truncated prism*.

(343.) The volume of a truncated triangular prism is equal to the sum of volumes of three pyramids whose base is one of the bases of the truncated prism, and whose vertices are at the three angles of the other base.

Draw a plane through the edge M O of the base M N O, and through the angle N′; this plane will cut off from the truncated prism a pyramid having for its base the base M N O, and for its vertex the angle N′. Draw another plane through the edge M′ N′, and through the angle O; this will cut off another pyramid having M′ N′ O′ for its base, and O for its vertex. The remainder of the truncated prism will be the pyramid whose base is M M′ N′, and whose vertex is O. But this will be equal to the pyramid which has the same base and its vertex at O′; because O and O′ are equally distant from the plane M M′ N′. Hence it follows that the volume of the truncated prism is equal to the two pyramids which have M′N′O′ for their common base and their vertices at M and O, together with the volume of the pyramid which has M N O for its base and its vertex at N′. But if the line N O′ be drawn, the pyramids whose common base is M N N′ and whose vertices are O and O′ are equal; and if N M′ be drawn, the pyramids whose common base is N N′ O′ and whose vertices are at M and M′ will have equal volumes. It follows, therefore, that the pyramid which has M N O for its base and its vertex at N′, will be equal to that which has M′ N′ O′ for its base and its vertex at N. Hence it appears that the whole volume of the truncated triangular prism is equal to the sum of the volumes of three pyramids which have M′ N′ O′ for their base, and their vertices at the points M, N, and O, which form the angles of the other base.

(344.) Since pyramids having equal bases and altitudes have equal volumes, it follows that the volume of a triangular truncated prism is equal to the sum of the volumes of three pyramids having one of the bases of the prism for their base, and having their altitudes

equal to perpendiculars drawn upon the one base of the prism from the three angles of the other base.

(345.) Let $M'' N'' O''$ be a section of the prism by a plane perpendicular to its edges. The volume of the truncated prism whose base is $M'' N'' O''$, and whose superior base is $M N O$, will then be equal to the sum of the volumes of three pyramids upon the base $M'' N'' O''$ whose vertices shall be M, N, and O; or, since the edges of the prism are perpendicular to $M'' N'' O''$, it will be equal to the sum of the volumes of three pyramids upon the base $M'' N'' O''$ with the altitudes $M'' M$, $N'' N$, and $O'' O$.

For the same reasons the volume of the prism on the base $M'' N'' O''$, and having for its superior base $M' N' O'$, will be equal to the sum of the volumes of three pyramids whose common base is $M'' N'' O''$, and whose altitudes are respectively $M'' M'$, $N'' N'$, and $O'' O'$. The difference between the volumes, therefore, which is in fact the volume of the truncated prism whose bases are $M N O$ and $M' N' O'$, is equal to the difference between the sum of the volumes of the three former pyramids having $M'' N'' O''$ as their common base, and the sum of the volumes of the three latter pyramids having the same common base, which difference will be equal to the sum of the volumes of three pyramids having the same common base $M'' N'' O''$, and the difference of the altitudes respectively of the two systems of pyramids as their altitudes, which differences will be $M M'$, $N N'$, and $O O'$.

It follows, therefore, that the volume of any triangular truncated prism whatever will be equal to the sum of the volumes of three pyramids whose common base is a rectangular section of the prism, and whose altitudes respectively are equal to the three edges of the truncated prism.

(346.) Since the volumes of prisms and pyramids having equal bases are proportional to their altitudes, it follows that the sum of the volumes of any number of prisms or pyramids having equal bases will be equal

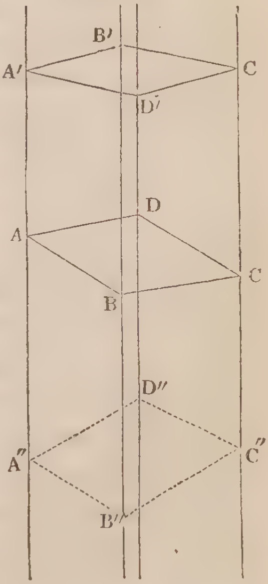
to the volume of one prism or pyramid having the same base, and whose altitude shall be equal to the sum of their several altitudes.

(347.) Since the volumes of prisms and pyramids having equal altitudes are proportional to their bases, it follows that the sum of the volumes of several prisms or pyramids having equal altitudes, is equal to the volume of one prism or pyramid with the same altitude, and whose base is equal to the sum of their several bases.

(348.) Since the volume of a truncated triangular prism is equal to the sum of the volumes of three pyramids whose common base is the rectangular section of the prism, and whose altitudes respectively are its three edges, it is equal to the volume of one pyramid whose base is the same rectangular section of the prism, and whose altitude is the sum of the three edges.

(349.) Let A B C D and A′ B′ C′ D′ (*fig.* 160.) be the bases of a quadrangular truncated prism whose faces are perpendicular to each other, and let A″ B″ C″ D″ be a rectangular section of it; let its volume be divided by two diagonal planes, one passing through the edges A A′, C C′, and the other through the edges B B′, D D′: the volume of the truncated triangular prism whose bases are A B D and A′ B′ D′ is equal to the volume of a pyramid whose base is A″ B″ D″, and whose altitude is the sum of the edges A A′, B B′, and D D′. In like manner the volume of the truncated triangular prism whose bases are B C D and B′ C′ D′ is equal to the volume of a pyramid whose base is B″ C″ D″, and whose altitude is the sum of the edges B B′, C C′, and D D′; therefore the volume of the quadrangular truncated prism

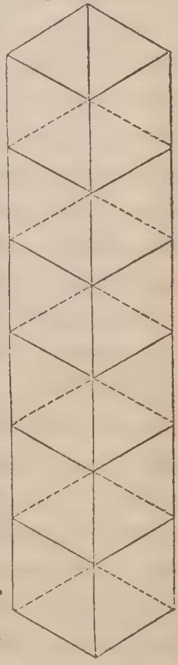
fig. 160.



is equal to that of a pyramid whose base is half the rectangular section $A''B''C''D''$, and whose altitude is the sum of the edges $A''A'$ and CC', together with twice the sum of the edges BB' and DD'.

In like manner it may be shown that the volume of the truncated quadrangular prism is equal to the volume of a pyramid whose base is half the rectangular section, and whose altitude is equal to the sum of the edges BB' and DD', together with twice the sum of the edges AA' and CC'; therefore twice the volume of the quadrangular prism will be equal to a pyramid whose base is half its rectangular section, and whose altitude is three times the sum of its four edges. The volume, therefore, of the quadrangular truncated prism will be equal to that of a pyramid whose base is a fourth part of its rectangular section, and whose altitude is three times the sum of its four edges. It is evident, therefore, that the volume of any truncated quadrangular prism of this kind, is equal to the volume of a rectangular parallelopiped whose base is the rectangular section of the prism, and whose altitude is the fourth part of the sum of its four edges.

fig. 161.



(350.) As the areas of all surfaces are expressed and calculated numerically by resolving them into the squares of the line taken as the linear unit, which square is therefore the superficial unit; so the volumes of all solids are expressed and investigated numerically by resolving them into cubes whose side is the linear unit, which cube is therefore the unit of volume, or the solid unit.

(351.) If the base of a rectangular parallelopiped (*fig.* 161.) be the square of the linear unit, its volume will consist of as many cubes of the linear unit as there are linear units in its height. In fact, it will be a square pillar, composed of a number of cubes of the linear unit placed one above

the other, and its volume will be expressed numerically by the number which expresses its height. Thus, if the base of the column be a square inch, and its height be ten inches, its volume will be ten cubic inches.

(352.) If the sides of the base of any rectangular parallelopiped be resolved into linear units, and the base itself by drawing parallels to its sides be resolved into squares of the linear unit, the number of such squares composing the base will be found, as has been already shown, by multiplying together the numbers expressing the sides of the base. From the angles of each of the squares into which the base is thus resolved, perpendiculars may be raised and continued till they meet the superior base of the parallelopiped. These perpendiculars will be the edges of columns of cubes of the linear unit of which the volume of the parallelopiped is composed, and there will be as many such columns as there are squares of the linear unit in the base of the parallelopiped: each column will contain as many cubes of the linear unit as there are units in the height of the parallelopiped. The volume of the parallelopiped will therefore be obtained numerically by multiplying the number of squares in its base by the number of units in its height; and since the number of squares in its base is obtained by multiplying together the numbers expressing the sides of the base, it follows that the number of cubical units composing the volume of the parallelopiped will be found by multiplying together the three numbers expressing the lengths of its three edges.

Thus, if the sides of the base of a rectangular parallelopiped be eight inches and nine inches, the area of its base will be 72 square inches; and if its height be ten inches, its volume will be 720 cubic inches.

(353.) Since the volume of any prism, whether right or oblique, and whatever be its base, is equal to that of a rectangular parallelopiped having an equal base and altitude, it follows that the volume of a prism is obtained numerically by multiplying the number ex-

pressing its altitude by the number expressing the area of its base.

(354.) Since the volume of a pyramid, whatever be the form of its base, is equal to one third of the volume of a prism with an equal base and altitude, it follows that the volume of a pyramid is found numerically by multiplying the number expressing one third of its altitude by the number which expresses the area of its base; or, what is the same, by multiplying the area of its base by one third of its altitude.

(355.) From what has been proved in (349.), it follows that the area of a truncated quadrangular prism whose perpendicular section is a rectangle, may be calculated numerically by multiplying the area of such section by the fourth part of the sum of its four edges.

(356.) This geometrical principle is applied in the calculation of the tonnage of ships.

The vessel, considered as a geometrical solid, is divided by horizontal planes at equal distances one above the other, and also by vertical planes equally distant in the horizontal direction. The whole capacity of the vessel is thus resolved into truncated prisms having equal rectangular sections, and whose bases will be determined by the form of the vessel. If the rectangular section of such prisms be expressed numerically by taking the square of the distances between the planes by which the vessel is divided, and such section be multiplied by the fourth part of the sum of the four edges of each prism, the number of cubical units corresponding to each prism will be found, and the addition of these will give the whole tonnage of the vessel.

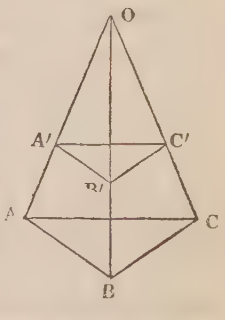
(357.) The volume of solids of every form may be calculated numerically by resolving them into pyramids. If a point be taken within the solid, and from it perpendiculars be drawn upon the several faces, the number expressing the area of each face multiplied by one third of the number expressing the length of the perpendi-

cular upon that face will give the volume of the pyramid whose base is that face and whose vertex is at the assumed point, and the sum of the numbers expressing the volumes of the several pyramids thus obtained will express the volume of the solid.

(358.) The preceding method of calculating numerically the volumes of solids is sometimes attended with difficulties in practice; and the method of truncated prisms, shown by its application to the determination of the tonnage of vessels, offers generally greater facility. Every solid may be resolved into truncated prisms by being supposed to be cut by two systems of parallel planes at right angles to each other, and the measurement and calculation of such prisms supplies an easy method of determining the volume of the solid.

(359.) If a triangular pyramid be cut by a plane parallel to its base, another pyramid will be formed whose edges will be proportional to the corresponding edges of the given pyramid, and the triangular faces of the two pyramids will be similar each to each.

fig. 162.



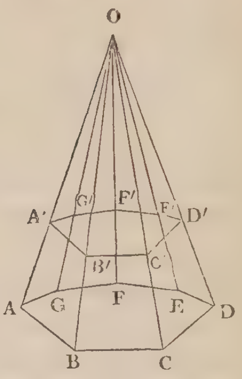
Let O (*fig.*162.) be the vertex of the pyramid, and let A′ B′ C′ be the section parallel to the base; it is evident that the triangle A′ O B′ will be similar to the triangle A O B, since A′ B′ is parallel to A B; and in like manner the other faces of the one pyramid will be similar to those of the other. The sides of the triangle A′ B′ C′ will be respectively proportional to those of the triangle A B C, being in the common ratio of the edges of the two pyramids; therefore the triangles A′ B′ C′ and A B C will be similar.

(360.) Two pyramids, such as here described, are said to be similar one to the other.

(361.) In like manner it may be proved that a plane parallel to the base of any pyramid, such as A B C D E F G (*fig.* 163.), will cut off a similar pyramid.

(362.) The volumes of similar pyramids are proportional to the cubes of their corresponding edges.

fig. 163.



For their bases being similar figures are proportional to the squares of their corresponding edges, and their altitudes being equally inclined to their edges are proportional to them; therefore their bases multiplied by their altitudes, which are three times their volumes, are proportional to the cubes of their corresponding edges. Their volumes, therefore, are proportional to the cubes of their corresponding edges.

(363.) Similar solids in general are those which consist of the same number of edges inclined at equal angles, and proportional each to each in length; the solids having the same number of faces, and these faces being similar each to each.

(364.) If two points be taken in corresponding positions within similar solids, these solids will be resolved into the same number of pyramids, which shall be similar each to each, by lines drawn from the assumed points to the several angles of the solids. The volumes of each pair of these similar pyramids will be proportional to the cubes of the corresponding edges of the solids, and therefore the solids themselves will be proportional to the cubes of their corresponding edges.

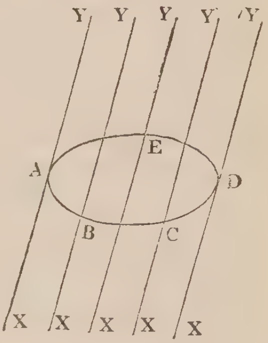
(365.) It is evident, then, that if the magnitude of any body be increased or diminished by the increase of its linear dimensions, the increase of its solid capacity will be much greater than that of its linear dimensions. Thus, if the height be doubled, all the other dimensions being likewise doubled, the solid dimensions will be increased in an eight-fold proportion; if the height and all the other dimensions be trebled, the solid dimensions will be increased twenty-seven-fold, and so on.

CHAP. XV.

OF CYLINDRICAL SURFACES.

(366.) Let ABCDE (*fig.* 164.) be a plane curve, and XY be a straight line passing through any point A in it, and inclined at any angle to its plane: if this line be supposed to move round the curve so as to be constantly parallel to itself, the surface which it describes as it moves is called a *cylindrical surface;* and the curve ABCDE, which governs the motion of the line XY, is called the *generatrix* of the cylinder. The line XY, by the motion of which the cylinder is thus produced, taken in any given position, is called the *side* of the cylinder.

fig. 164.



(367.) If the moving line be perpendicular to the plane of the generatrix, the cylinder is called a *right cylinder ;* and if it be oblique to that plane, it is called an *oblique cylinder.*

(368.) It is evident that a plane surface, in a general sense, belongs to the family of cylindrical surfaces ; for if the generatrix ABC be a straight line, the surface produced by XY will be a plane.

(369.) If the generatrix be a right-lined figure, it is evident that the line XY will produce a prism. The prism and cylinder, therefore, belong to the same class.

(370.) Cylindrical surfaces may likewise be produced by the motion of any plane figure, ABCDE, parallel to itself along a fixed straight line XY. As in the former case all the points of the moving line XY described figures in parallel planes equal and similar to the gene-

ratrix; so, in the present case, all the points of the generatrix describe straight lines equal and parallel to that along which the point A is moved.

(371.) From either mode of generating a cylindrical surface, it is evident that all sections parallel to the generatrix are figures equal and similar to the generatrix, and all sections by planes through the sides are straight lines.

(372.) There is no form of body so constantly required in the arts as the various family of cylindrical surfaces, and the methods resorted to for their production are based on one or other of the principles above described. Let us call, for distinction, the right line which measures the length of the cylindrical surface its directrix; it is evident that a straight edge applied to such a surface parallel to the directrix will touch it in every part, while its section by a plane parallel to the generatrix will always be a figure equal and similar to the generatrix itself.

There are then in practice four processes by which a cylindrical surface may be formed.

1. A straight edge representing the directrix may be moved over a figure representing the generatrix, and as it moves it may reduce the surface of the body to the required cylindrical form by cutting, pressing, or by the production of any other mechanical effect capable of changing the form of the body.

2. A straight edge representing the directrix may be maintained in a fixed position, and the body to which the cylindrical form is to be imparted may be moved in contact with it in accordance with the figure of the generatrix. As it moves, the straight edge will, as before, give it the required form.

3. An edge being constructed in the form of the generatix may be moved along another edge representing the directrix, and as it moves against the body to which the cylindrical form is to be imparted it will give the desired form to that body.

4. The same edge or surface having the form of the

generatrix may be fixed, and the body to which the cylindrical form is to be imparted may be moved in contact with it along a straight edge representing the directrix, and as it moves it will receive the required cylindrical form.

(373.) The process of wire-drawing is one in which a cylindrical form, with a circle for its generatix, is required to be imparted to the metal of which the wire is made. A hole corresponding in magnitude is formed in a plate of hardened steel; and the metal of which the wire is to be formed, being at first a little thicker than this hole, is forcibly drawn through it, and is thus reduced by pressure to the required magnitude. When the thickness of the metal is to be considerably reduced, a succession of these holes, gradually diminishing in magnitude, are made in the same steel plate, and the wire is drawn successively through them, being thus gradually reduced to the proper dimensions. This process corresponds to the production of a cylindrical surface, by the motion of the generatrix parallel to itself along the directrix.

(374.) In general this method of producing cylindrical surfaces is resorted to in cases where, like that of wire, the length of the cylinder is very considerable in proportion to its thickness; but the same process is sometimes resorted to where a great number of cylinders precisely equal and similar are required to be produced, having their length extremely small in proportion to their diameter or breadth. An example of this is presented in the manufacture of the wheels and pinions used in watchwork. The external form of these is that of a circle serrated at its edges, with projecting teeth formed with great precision and equality throughout the circumference. The wheel is a cylinder whose generatrix is such a serrated circle, but whose height or thickness is exceedingly small in proportion to its diameter. If each wheel were fabricated by a separate process, the expense of the manufacture would be excessive. Instead of this, an aperture is formed in a plate

of hardened steel, to which the exact form of the generatrix of the wheel is imparted. A rod formed of the metal of which the wheels are to be made, being very nearly equal to the aperture, is then forced through it; and the cylindrical surface is produced, of which the contour of the aperture in the steel plate is the generatrix. This surface is fluted with ridges corresponding exactly in form and magnitude to the teeth of the wheel. It is then cut in slices perpendicular to its length, corresponding to the thickness of the wheel; and a vast number of wheels are produced precisely identical in form and magnitude.

(375.) By a process nearly similar to the preceding, cylindrical or prismatical forms of various kinds are imparted to iron for various purposes in the arts: the bars, for example, which form iron railways, are thus produced. Two rollers of hardened steel are firmly fixed in axles or bearings parallel to each other, and so that the surfaces of the rollers are nearly in contact. The faces of these rollers are so formed that an aperture is left between them as they turn, corresponding in form and magnitude to the generatrix of the cylinder or prism which it is desired to produce. A lump of iron rendered white hot in a furnace, and therefore in a soft state, is then taken, and being submitted to the blows of a heavy hammer is reduced to the form of a rod of sufficient length, and of dimensions corresponding nearly to the aperture between the rollers. The rollers being kept in a state of revolution by a steam engine or other moving power, one end of the bar of iron, still in its red and soft state, is presented to the aperture between the rollers, and being pinched by them is drawn in between them as they revolve, and is discharged at the other side, having received the form corresponding to the aperture in the rollers. A succession of apertures gradually diminishing in magnitude, but similar in form, is usually provided between the same pair of rollers; and the bar of iron, while still red and soft, is transferred successively from side to side through these apertures till it is reduced to the proper

magnitude. The rude lump of red iron is thus reduced to a finished bar or rail in a space less than a minute, and without requiring to be reheated.

In the same manner iron rods of every form, and of all dimensions, are constructed by the process of rolling. Sheet iron is similarly produced by rollers whose surfaces are perfectly flat; the metal being passed successively between different pairs of rollers, gradually decreasing in their distance one from the other.

(376.) When the material of which the cylindrical surface is to be formed is wood, the process of *cutting* must generally be substituted for that of drawing or *rolling*. A cutter is usually formed into the figure of the generatrix of the cylinder, and being fixed in a frame by which it can be guided in its motion along the directrix, it is passed over the surface of the wood to which the cylindrical or prismatical form is to be imparted. Such an instrument is called a *plane*. It is by such a tool that all mouldings are formed in carpentry. It has been already stated that in a general sense a plane surface belongs to the family of cylinders. We accordingly find that such a surface is produced in carpentry by the same class of tools as is used for the production of mouldings, the cutting edge being straight when a plane is required to be produced.

(377.) When the cylindrical surface required is of great magnitude, the application of this class of tools sometimes becomes impracticable. In that case, if great accuracy in the section of the cylinder perpendicular to its directrix be not required, it may be approximately formed by the motion of a plane-cutting tool parallel to its directrix, the position of the tool being constantly shifted according to the form of the generatrix: it is in this manner that the masts of ships are formed.

(378.) When the last degree of precision is required in the cross section of the cylinder as well as in the direction of its length, the lathe is the instrument resorted to. The substance to which the cylindrical form is to be imparted is placed between the centres of the

lathe, and a motion of revolution is given to it; the point of the cutting tool, being fixed in its position, is then presented to it, and as the body revolves, it cuts off from it all those parts which project beyond the proper distance from its centre; and this process is continued until that part of the body acted on by the tool is reduced to the proper form. The tool being fixed upon a guide, by which it can be moved parallel to the directrix of the cylinder, is then shifted in its position, and another part of the cylinder is formed; and this process is continued until the cylinder is completed.

(379.) A circular cylinder of wood is sometimes formed by forcing the wood through a circular cutter or plane.

(380.) When the body to which the cylindrical form is to be given is too massive to be made to revolve with convenience, the motion of revolution is given to the cutter, the body remaining fixed. In this manner the interior surfaces of great steam cylinders are formed. Being reduced by casting to nearly the proper form, a cutter is made to revolve within them in close contact with their surfaces; and while it revolves a slow progressive motion is given to it, so that it is made to pass gradually from end to end of the cylinder.

(381.) When the last degree of precision is not required to be given to the surface, and when the material is capable of fusion, the process of casting is the most expeditious and cheap method of forming cylinders. A pattern of the cylinder is accurately formed in wood, and from that a mould is taken in sand, plaster, or other convenient material. The molten metal is poured into such mould; and being allowed to harden by becoming cold, the sand or plaster is removed, and the cylinder is obtained.

(382.) When the material is soft and capable of fusion at low temperatures, a permanent mould of metal is used, from which the cylinder, after being cast, is drawn; the same mould constantly serving for the reproduction of other cylinders. In this manner the ma-

nufacture of candles is conducted. A mould of metal is constructed, having the exact form of the candle, the inner surface of which is reduced to a high polish; and the wick is stretched along its axis, leaving a loop at one end, across which a rod of wood or metal is extended. The liquid grease or wax is then poured into the mould, and when it has hardened by cooling it is drawn out by means of the rod of wood or metal.

(383.) From the method in which a cylindrical surface has been described to be produced, it is evident that a plane surface may be reduced by flexure to the form of any cylindrical surface whatever. On this principle cylinders are formed in the arts by bending thin plates of metal, and sometimes even of wood, into the proper form: plates of tin or sheet iron, being bent into the circular form, and united at their edges by soldering, form the chimneys of stoves, the gutters of houses, &c.

Various vessels used in domestic economy, especially for culinary purposes, receive the cylindrical form by the same means. The boilers of steam engines are usually in the cylindrical form, the generatrix varying very much in figure, according to the circumstances under which the boiler is to be used.

(384.) In the application of the arts to the purposes of science a combination of minuteness and accuracy of construction is sometimes required, the attainment of which demands peculiar methods. In the construction of astronomical telescopes, the space formed by what is called the field of view is partitioned out by a system of parallel threads or wires extended across it: these wires must be of such accurate construction, and so minute in size, that when seen with the high magnifying power used in these instruments they shall still appear to be lines accurately straight, and so small in breadth that they shall appear to the eye like a fine hair. Such wires, when presented to the naked eye, would be scarcely if at all visible. Yet they require to be con-

structed truly cylindrical. The process of constructing these wires, invented by the late Dr. Wollaston, was as follows: — A cylindrical mould being formed, a thread of gold or platinum wire is extended along its axis in the same manner as the wick is extended along the mould of a candle; another ductile metal which melts at a lower temperature being fused, is then poured into the mould, and a small cylinder of metal is thus produced, having the thread of gold wire in its axis. This cylinder is then submitted to the process of wire-drawing, until it is reduced to a great degree of tenuity. Throughout this process the thread of gold wire is still extended through its axis, being itself wire-drawn with the cylinder in which it is enclosed, and its thickness still maintaining the same proportion to that of the cylinder. When the process of wire-drawing has been completed, the compound wire is exposed to the action of an acid, by which the external metal is dissolved, but which cannot attack the thread of gold wire extended along its axis. The fine gold wire is thus stripped of its coating; and being extended across the field of view of the telescope, serves the purposes above mentioned.

By this process threads of gold wire may be formed, 10,000 of which, placed side by side, would not cover more than an inch.

(385.) The species of cylinder of most common occurrence in the arts is that whose generatrix is a circle; and the most common of this species is the right cylinder: the use of this is so frequent, compared with any other form of cylinder, that the term *cylinder*, except in the higher mathematics, is always understood to express the right circular cylinder; and it will be here so used, unless otherwise expressed.

(386.) The generatrix limiting the length of a cylinder, and forming its plane circular ends, is called its *base*. A straight line joining the centres of the bases of a cylinder is called the *axis* of a cylinder.

(387.) All sections of a cylinder by planes perpendicular to its axis are circles equal to its bases.

(388.) All sections of a cylinder by planes parallel to its axis are parallelograms.

(389.) As the surface of a cylinder may be formed by bending a plane surface according to the form of the generatrix of the cylinder, it is evident that the surface of a right cylinder, whatever be the nature of its generatrix, will, if unbent or unfolded so as to be spread out into a plane, be a rectangle, whose height is the height of a cylinder, and whose base is the perimeter, or circumference of its base: the area of the convex surface, therefore, of a right cylinder will be found by multiplying its height by the circumference of its base; and this will be equally true whatever be the generatrix.

(390.) The area of the sides of a right prism is, for the same reason, found by multiplying its height by the perimeter of its ends or bases.

(391.) If a cylinder be oblique, its convex surface, if spread out into a plane, will form an oblique parallelogram; and the same will be true of an oblique prism or a cylindrical surface, whatever be its generatrix.

(392.) The area of the curve surface of a cylinder, or the sides of a prism, whether right or oblique, will, therefore, be found by multiplying the perimeter or circumference of its base by the perpendicular distance between the parallel planes that form its ends.

(393.) The above calculation of cylindrical surfaces does not include the areas of their bases. Since the area of a circle is equal to half the rectangle under the radius and circumference, the areas of the circular ends of a cylinder will be equal to the rectangle under the radius and the circumference of the base; therefore, the area of the whole surface of a circular cylinder, including its ends, will be equal to the rectangle under a line, equal to the sum of its altitude and the radius of its base, and the circumference of its base; or, what is the same, its total surface will be found numerically by adding to its height the radius of its base, and multiplying the sum by the circumference of its base.

(394.) If the volume of a cylinder be considered, like that of a prism, to be composed of a number of equal plates laid one over another, it is evident that it will be equal to the volume of a prism whose base is of equal area, and which has the same altitude; for the volumes of such solids will be composed of the same number of plates of equal magnitudes. A prism and cylinder, therefore, having equal bases and equal altitudes, have equal volumes.

(395.) The volume of a cylinder will be found numerically by multiplying the area of its base by its altitude.

(396.) While the base of a cylinder remains the same, its volume will increase or diminish in the same proportion as its altitude is increased or diminished; and while its altitude remains the same, its volume will increase or diminish in the same ratio as its base increases or diminishes.

(397.) The preceding properties of the volumes of cylinders equally belong to every cylinder, whatever be its generatrix, and whether it be right or oblique.

(398.) The volumes of circular cylinders are proportional to their heights multiplied by the squares of their diameters, because the areas of their bases are proportional to the squares of their diameters.

(399.) The determination of the shadows produced by the light of the sun falling upon opaque objects involves the properties of cylindrical surfaces. The rays of solar light proceeding in parallel lines, a part is intercepted by the opaque body; but those rays which pass immediately beyond its edges proceed in parallel lines till they reach the surface on which the shadow is projected, where they mark the boundary between the illuminated part of the surface and the shadow, or that part which is deprived of light by the interposition of the opaque body. The rays of light, therefore, which thus touch the edges of the body, form a cylindrical surface, of which one base is a section of the body which projects the shadow made by a plane perpendicular to the rays of light, and the other base is the shadow itself. The determination of shadows thus depending essentially on

the properties of cylindrical surfaces, this part of geometry is necessary to the right understanding and practice of architecture, painting, and those arts of design in which the effects of lights and shadows are to be investigated or represented.

(400.) The position and form of lines in space are expressed, in the higher geometry, by determining the projection of these lines on planes placed at right angles to each other. Two such projections being given, the line in question will be perfectly known.

From every point of the line whose form and position are to be determined let perpendiculars be supposed to be drawn to a horizontal plane, such as the floor of a room; these perpendiculars will form a cylindrical surface, of which the line in question is the generatrix or base. The points of the horizontal plane where the perpendiculars meet it will form the horizontal projection of the line, and will be the other base of the cylinder. If perpendiculars be in like manner drawn from the line to a vertical plane, such as one of the walls of a room, they will form another cylindrical surface, of which the line is also the base or generatrix; and another projection of it, forming the other base of the cylinder, will be formed on the vertical plane.

If these two projections, one on the horizontal and the other on the vertical plane, were given, the line of which they are the projections would be found by constructing two cylindrical surfaces, having these two projections as their bases, and perpendicular respectively to the two planes on which the projections are given. The line formed by the intersection of these two cylindrical surfaces would be the line sought.

(401.) If a cylinder be laid with its side upon a plane, the points at which it will meet the plane will lie in a straight line, forming the side or one of the positions of the directrix of the cylinder. All other points of the plane will lie outside the cylinder. The plane is in this case a tangent plane to the cylindrical surface. If the cylinder be rolled upon the plane, each line of

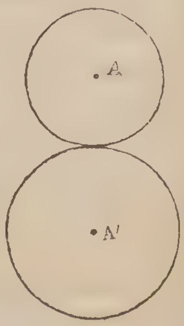
contact which it assumes with the plane will be parallel to all former lines of contact. In fact, the line of contact of the cylinder with the plane will move parallel to itself, and will be parallel to the axis of the cylinder, which likewise moves parallel to itself.

If the cylinder be a right circular cylinder, its axis will, during such motion, move in a plane parallel to that on which the cylinder rolls, and at a distance above it equal to the radius of the cylinder.

(402.) The form of a plane is imparted to soft substances by virtue of this property of the right circular cylinder. In agriculture, when the surface of a tilled field is required to be made plane by breaking or pressing down the rough mould which the plough or harrow has left upon it, and in gardening, when the rough surface of loose gravel forming a walk or road is required to be rendered even and plane, a heavy cylindrical roller of iron or stone is passed over it, which, forcing itself into contact by its weight with the surface on which it rolls, reduces that surface to the plane form, without which continued contact with it would be impossible. Since, however, a cylindrical roller passing in *one direction* only will not produce a level surface, in the formation of a plane where great precision is required the roller should be passed over frequently and in *various directions*.

(403.) If two right circular cylinders be placed with their axes parallel one to the other, and so that the distance A A′ (*fig.* 165.) between the axes shall be equal to the sum of their radii; then the surfaces of these cylinders will touch each other, and their line of contact will be a straight line parallel to their axes, being, in fact, a side of the cylinder.

fig. 165.



If two cylindrical surfaces thus placed be intersected by a plane at right angles to their axes, their section by that plane will be two circles equal to

the bases of the cylinders, which will touch each other externally, as represented in *fig.* 165.

(404.) If one of the cylinders, A, thus placed be rolled upon the other, their line of contact will move parallel to itself, being always a common side of the two cylinders; and the axis A of one cylinder will move parallel to itself round the axis of the other, describing the surface of a right circular cylinder, whose radius A A′ is equal to the sum of the radii of the two given cylinders, and whose axis is the axis A′ of the fixed cylinder.

(405.) If the surfaces of two cylinders thus placed in contact and pressed together be so rough that one cannot move without moving the other with it, and that both be capable of revolving upon their axes, then any motion of revolution which is given to one cylinder will be imparted to the other, the surfaces of the two cylinders moving at the same rate.

(406.) It is on this principle that wheel work in machinery acts. The moving power, whatever it may be, gives motion to one wheel or cylinder, the edge of which, pressing on another, imparts motion to it, and that again acts on another, and so on. As the actual velocity of the edges of the wheels in contact will be the same, the velocities of revolution are varied by varying the magnitudes of the wheels. If the diameter of the wheel A (*fig.* 165.) be half the diameter of the wheel A′, then it will require two revolutions of the former to produce one of the latter, and the velocity of revolution of the former wheel will be double that of the latter. In fact, the velocities of revolution of each pair of contiguous wheels will be in the inverse proportion of their diameters.

(407.) If the surfaces of the cylinders thus in contact were perfectly smooth, the revolution of one upon its axis would not impart motion to the other, but the surface of the one would slide on that of the other. In proportion to the roughness of the surfaces, friction will

be produced between them; and the resistance attending this friction will cause the surface of the second cylinder to be pushed round, and that the one cylinder, instead of sliding, shall roll upon the other. If so great a resistance, however, be opposed to the motion of the second cylinder as to exceed that produced by the friction of the surfaces, then, notwithstanding the friction, the surface of the one will still slide upon the surface of the other without imparting motion to it. In this case the resistance due to friction is increased either by coating the surfaces of the cylinders with leather, or some other rough material; or if they be wood, by cutting them with their grains in contrary directions. But where the resistance is considerable, or where the inaccuracies of motion produced by the occasional and accidental slipping of one surface on another must be avoided, as in the case of watchwork, then the surfaces are formed into teeth, of equal and uniform magnitude and form, which insert themselves between one another, and render any inequality of motion impossible, unless by the fracture of a tooth.

(408.) Whatever be the surface in contact with which a right circular cylinder is rolled, its axis will move in a parallel surface; and the same will be true of whatever may be supported by such an axis. A wheel carriage moving along a road is therefore carried in lines parallel to the road; since the wheels are right circular cylinders in contact with the road. Hence it is that all inequalities of the road produce corresponding inequalities of motion in every part of the carriage.

(409.) If a cylinder be in contact with any surface on which it is prevented from sliding by the resistance attending its friction with it, and if at the same time the surface on which it is placed be fixed and incapable of moving under it; then any motion of revolution which may be imparted to the cylinder must, at the same time, give to the cylinder a progressive motion

along that surface. For as the surface of the cylinder is prevented from rubbing or slipping on the surface on which it rests, it cannot turn round except by rolling on that surface; and it cannot roll on that surface without advancing along it with a progressive motion.

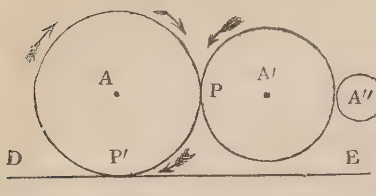
Thus, if any force be applied to the spokes of the wheels of a carriage, so as to compel the wheel to turn round, and if by the pressure of the wheel upon the road it is prevented from slipping as it revolves; then the carriage must roll onwards by the revolution of the wheels, in the same manner as if it were drawn forwards in the common way by horses or any other tractive power.

(410.) It is on this principle that the steam engine is applied to produce the progressive motion of carriages upon railways. The wheels of the engine are fixed upon their axle, so as to turn with it, and not upon it, as in common carriages. On the axle of these wheels is formed a crank or arm like the handle of a winch or windlass. The piston-rod of the steam engine lays hold of this arm, and as the piston is driven backwards and forwards in the cylinder causes the arm to revolve. As the arm revolves the axle on which it revolves also revolves, and with this axle the wheels fixed upon it are made to revolve. Now, these wheels resting upon the rails, with the incumbent weight of the engine upon them, must, as they revolve, either slip on the rails or roll forward, causing the engine to roll with them; and as the resistance produced by their pressure upon the rails is so great as to prevent their slipping, the engine is compelled to roll forwards, and to draw after it the train of carriages or waggons.

(411.) In the modern printing presses the properties of a cylinder moving in contact with a plane is brought into frequent operation. In letterpress printing the stereotype plates, having upon their faces in relief the letters to be printed on the paper, are bent so as to correspond to the form of a large cylinder or roller to which they are attached. Another cylinder or roller is

placed in external contact with this, as represented in *fig.* 166. Let A be the cylinder on the surface of which the letters to be printed are placed, and let D E be a plane in contact with it, on which the paper is extended: as the cylinder turns in the direction of the arrows, the surface E D advances under it in the same direction, and the types thus brought into successive contact with the paper leave their impressions upon it. The roller A′ contains on its surface a quantity of ink, which is spread upon it evenly and uniformly by the roller A″, with which it is likewise in contact, and which latter roller is supplied from a reservoir of ink with which it communicates. As the types pass the point of contact P of the rollers A and A′, they receive the ink from the surface of the roller A′; and as they pass the point of contact P′ of the roller A with the paper, they leave the ink in the form of the letters upon the paper, and they are carried round again to the point P to receive a fresh supply of ink for the next impression.

fig. 166.



(412.) This method of cylindrical printing is subject to defects which are inadmissible in the better class of presswork, and indeed has been discontinued even in the *cheaper* description of printing in England. The cylinders in newspaper printing are still used, but they carry the paper and not the types. The types are set or laid in a plane surface, and are moved under the cylinder on which the paper is rolled, and by which it is brought into contact with, and pressed upon, the type. Where great expedition is required, the paper is made to pass by means of cords or tapes successively over two or more cylinders, so as to be reversed in its position, and to have its opposite sides brought into successive contact with the types from which it is to receive the impression; each sheet is thus printed on both sides by the same operation.

(413.) With hand-presses, before the improvement of printing machinery and the application of steam to that branch of the useful arts, two hundred and fifty copies were obtained per hour from the same types, which required the work and superintendence of two men,—a cylindrical press worked by steam is now capable of printing two thousand sheets per hour on both sides, and requires only the attendance of two children.

(414.) The application of cylinders to calico printing forms one of the most important modern improvements in that branch of manufacture. Accurately formed cylinders of copper have their surfaces engraved with the pattern required to be impressed on the cloth. These cylinders or rollers revolve in contact with others, which are evenly smeared with a dye of the colour corresponding to that required for the pattern: as the copper cylinder passes that which contains the dye, it receives from it a coating of the colouring matter; it then passes in contact with a straight edge or scraper placed parallel to its axis. By this the colouring matter is wiped clean from the cylinder, except from the incisions upon it which mark the pattern to be printed. It is then rolled in close contact with the cloth, which is pressed against it by another cylinder, and which, as it passes, takes the colouring matter from the pattern engraved on the copper roller. In this manner a piece of calico of any length is printed by merely causing it to pass with a continuous motion between the rollers.

(415.) By the process here described the pattern would be printed only in one colour; but by a further improvement, the same principle has been applied for the production of patterns of two or more colours.

That part of the pattern which corresponds to each colour is engraved on a separate copper roller, and each roller is put in contact with another, from which it receives the proper colouring matter. The rollers are fixed in the same frame with their axes, parallel to and at such distances from each other, that, as the cloth passes under them successively, that part of the pattern

engraved on each roller falls in its proper place upon the cloth, so that the united effects of the several rollers is the production of a figure on the cloth, in which as many different colours are introduced as there are different rollers.

By such means it is not uncommon to witness the completion of the printing a piece of calico in three or four colours, in the space of thirty seconds.

(416.) It is not difficult to conceive the application of the same principle to the production of printed paper in several colours for the walls of rooms.

(417.) The application of cylinders to the manufacture of paper has produced a great improvement in that branch of art. Two cylinders having their axes parallel are placed nearly in contact, the distance between their surfaces corresponding to the thickness of the paper to be produced. As they revolve the matter of which the paper is fabricated passes between them, and sheets of any required length can be produced by a continuous motion of the cylinders.

(418.) In lithographic printing, the surface of a stone of very fine grain is reduced to an accurate plane by the process of grinding. On this surface the design to be printed is drawn; and being properly inked, the paper is pressed upon it by a cylinder rolled over the stone with great pressure.

(419.) Engravings on copper and steel are printed by passing the plate with the paper upon it between two cylinders placed with their axes parallel, and their sides in such near contact as to give the necessary pressure to the paper upon the engraved plate.

(420.) In every part of the art of spinning cotton numerous applications of the properties of cylinders are found.

The fibres of the raw wool are cleansed and arranged in parallel directions by the process of carding, which is conducted in the following manner:—A number of small wires are fastened in leather in a manner similar to the hairs which form a common brush. This leather

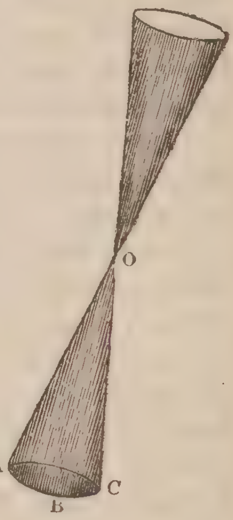
is attached to the surface of a large cylinder so as to form what might be called a large cylindrical brush with wire hairs. Two or more such cylinders are placed nearly in contact with one another, and moved with motions which are slightly different in speed; the raw cotton wool is spread in a box in contact with one of these cylindrical cards, which, as it revolves, carries away the wool spread upon the points of its wires. As this passes in contact with the other cylinder, the difference between their motions causes the one to rub upon the other like two brushes drawn one over the other: the wool is thus passed from the one to the other, and it is spread more thinly and evenly on the surface of the second card than it was on the first. By the repetition of this process the threads of the wool are at length arranged with the most perfect regularity, and it is subsequently collected into threads preparatory to the process of spinning or twisting.

CHAP. XVI.

OF CONES.

(421.) If a straight line A X (*fig.* 167.) passes through a fixed point O, and be moved through any curve such as A B C, it will trace as it moves a surface which is called a *cone.*

fig. 167.



(422.) The point O is called the *vertex* of the cone. The right line by the motion of which the surface is produced is called the *directrix* of the cone; and the curve A B C, which guides its motion, is called the *generatrix.*

(423.) If the generatrix of a cone be a right-lined figure, the cone will become a pyramid.

(424.) As all sections of a pyramid parallel to its base are figures similar to the base, whose linear dimensions are proportional to the distance of the section from the vertex; so in the same manner, and for like reasons, all sections of a cone parallel to the generatrix are figures similar to the generatrix, whose linear dimensions are proportional to their distances from the vertex.

(425.) The form of cone most commonly considered is that whose generatrix is a circle.

(426.) The *axis* of a cone is a line drawn from its vertex to the centre of its circular base.

(427.) A *right cone* is one whose axis is perpendicular to its base, and an *oblique cone* is one whose axis is oblique to its base.

(428.) If a cone and pyramid have equal bases and equal altitudes, their sections at equal distances from their vertices will have equal areas; for the linear dimensions of these sections and the bases being proportional to their distances from the vertices, the squares of these dimensions will bear the same ratio to the squares of the areas of the bases. The areas, therefore, of equidistant sections will be proportional to the areas of the bases; and the latter being equal, the former will be equal.

(429.) A pyramid and cone, therefore, having equal bases and equal altitudes, will have equal volumes; for, since all corresponding sections parallel to the bases are equal, the cone will be composed of a series of plates equal respectively to those which compose the pyramid.

(430.) The volume of a cone will be found, therefore, by multiplying the area of its base by one third of its altitude.

(431.) If a cone and cylinder have equal bases and equal altitudes, the volume of the cone will be one third of the volume of the cylinder.

(432.) The volumes of cones being proportional to the products of their bases and altitudes, and the bases being proportional to the squares of their diameters, the volumes will be proportional to their altitudes multiplied by the squares of the diameters of their bases.

(433.) Similar cones and cylinders are those whose altitudes are proportional to the diameters of their bases, and which, if oblique, have their axes equally inclined to their bases.

(434.) The volumes of similar cylinders and cones are proportional to the cubes of the diameters of their bases; for the areas of their bases are as the squares of their diameters, and the altitudes are as the diameters; therefore the altitudes multiplied by the squares of the diameters are as the cubes of the diameters.

(435.) If the base of a right pyramid be a regular polygon, its faces will all be equal isosceles triangles, whose bases are the sides of the polygonal base, and whose common vertex will be the vertex of the pyramid. If a

perpendicular be drawn from the vertex of the pyramid to one of the sides of the base, the area of the corresponding triangle will be equal to the rectangle under such perpendicular and half such side; and as the same will be true for each of the triangular faces, and as all the perpendiculars from the vertex on the sides of the base will be equal, the total area of the surface of the pyramid will be equal to the rectangle under such perpendicular and half the perimeter of the base.

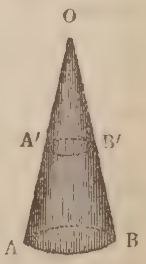
(436.) If the polygon forming the base of a pyramid have its sides successively both increased in number and diminished in magnitude, it will approximate to a circle, and the pyramid will approximate to a cone. Throughout such changes the area of the surface will still be equal to the rectangle under the perpendicular and half the perimeter of the base. If the sides then be conceived to be both indefinitely increased in number and diminished in magnitude, the base will become a circle, and the pyramid will become a cone; and the surface of the cone will accordingly be equal to the rectangle under the length of its side and half the circumference of its base.

(437.) The area of the surface of a right cone is therefore found by multiplying the length of its side by half the circumference of its base.

(438.) The area of the surface of a right cone is equal to that of a triangle whose base is equal to the circumference of the base of the cone, and whose altitude is equal to the side of the cone.

(439.) If a cone be cut by a plane A′ B′ (*fig.* 168.) parallel to its base, the figure having parallel circular bases thus cut off is called a *truncated cone;* and the area of its surface and its volume will be found by taking the differences of the surfaces and of the volumes of the whole cone A O B, and of the cone A′ O B′, which is cut off.

fig. 168.

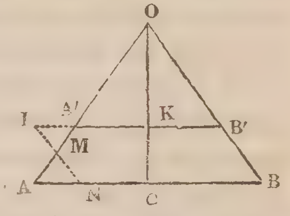


(440.) As the area of the whole conical surface A O B is that of a triangle whose height is A O, and whose base is the cir-

cumference of the circle A B, and the conical surface A′ O B′ is equal to a triangle whose height is A′ O, and whose base is the circumference A′ B′, it follows that the surface of the truncated cone will be equal to the difference between the areas of these triangles.

Let A B (*fig.* 169.) be equal to the circumference of the circle A B (*fig.* 168.), and from its middle point C draw a perpendicular C O equal to the side of the cone A B O, and join A O, B O; the area of the triangle A O B will then be equal to the surface of the cone A O B (*fig.* 168.). Draw A′ B′ parallel to A B, and at the same distance from O as A′ (*fig.* 168.) is from O; the area of the triangle A′ O B′ (*fig.* 169.) will then be equal to the surface of the cone A′ O B′ (*fig.* 168.). Hence it follows that the area of the surface of the truncated cone A A′ B′ B (*fig.* 168.) will be equal to the area of the trapezium A A′ B′ B (*fig.* 169.). Let A A′ be bisected at M, and through M let L N be drawn parallel to B B′; the triangle A M N will then be equal to the triangle L M A′, for in these two triangles the sides A M and A′ M are equal, and the angles are respectively equal. The areas of the triangles will, therefore, be equal (61.). The parallelogram B N L B′ will then be equal to the trapezium A A′ B′ B; because the parallelogram is formed by taking from the trapezium the triangle A M N, and adding to it the equal triangle A′ M L. But the area of the parallelogram is equal to its altitude C K multiplied by its base B N. Now the base B N is half the sum of the bases A B and A′ B′ of the trapezium, because B N is equal to B′ L, and the latter is equal to B′ A′, together with A′ L, or with A N, which is equal to A′ L; therefore, B N being equal to B′ A′, together with A N, must be equal to half the sum of A B and A′ B′. The area of the trapezium is, therefore, equal to its altitude C K multiplied by half the

fig. 169.



sum of its bases; and therefore the area of the surface of the truncated cone is equal to its side A A′ (*fig.* 168.) multiplied by half the sum of the circumferences of its bases.

(441.) The most accurate method of producing the form of a circular cone in the arts is by the lathe. While the body to which the conical form is to be given is kept in a state of constant revolution, the cutting tool is moved along the directrix or side of the cone. As it advances the circular form is given to the section of the body by its own motion, and the rectilinear form given to its side by the motion of the tool.

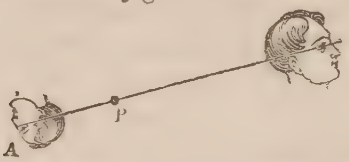
(442.) Of all the applications of the properties of cones in the sciences and arts, the most important and striking are those which have reference to the phenomena of light and vision. If rays of light proceed from a luminous point, diverging as they do in every direction, they always form a cone whose vertex is the luminous point, and whose base is the object they illuminate. If they fall on an opaque object, and a shadow of it be projected on any more distant surface, the shadow and the object will be the bases of a truncated cone, the vertex of which will be the luminous point. The shadow will in this case be greater than the object, and their linear dimensions will be proportional to their distances from the luminous point: thus, if the surface receiving a shadow be as far from the object which projects the shadow as the object itself is from the luminous point, the shadow will have twice the linear dimensions of the object. The surface on which the shadow is projected is here supposed to be parallel to the object. If it be not, the form and dimensions of the shadow will still be determined by the properties of the cone; for the shadow will still be the intersection of the cone of rays, whose vertex is the luminous point, and the object by which the shadow is projected, a section of the cone.

(443.) The *Lithouette* machine for taking profiles is constructed on these principles, being nothing more than

the shadow of the profile of the person whose likeness is required thrown upon a surface and there delineated.

(444.) Another method of taking likenesses in profile is founded still more immediately on the geometrical principle by which conical surfaces are produced, as described in (421.). A straight rod is fixed on a pivot

fig. 170.



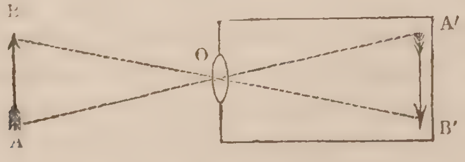
P (*fig.* 170.) so as to have free motion in every direction round it, and extending to some distance on both sides of it. At one end of the rod a pencil A is attached which moves over the paper that is to receive the likeness, while the other end is moved over the profile of the person whose likeness is to be taken, the pencil delineating the countenance in a reversed and inverted position, as represented at A in *fig.* 170.

If the pivot P be at one end of the rod, and the pencil A has any position between the two extremities, the countenance will be drawn in its natural position, as represented at A in *fig.* 171.

fig. 171.

(445.) In the instrument called the *camera obscura*, an object at A B (*fig.* 172.) is placed before a

fig. 172.



convex lens, which is fixed in the end of a close chamber, and the cone of rays of which the object is the

base, and whose vertex is at the centre of the lens O, falls on a surface beyond the lens, and produces an inverted picture A′ B′ of the object A B. The picture and the object form thus parallel bases of opposite cones.

(446.) The *camera obscura* is one of the feeble attempts of art to imitate nature. The eye is a camera obscura of exquisite perfection and sensibility. In front of the sphere which forms the eyeball is the circular opening called the *pupil,* which produces the black circular spot seen in the centre of the *iris,* or coloured membrane of the eye. Immediately behind this opening is suspended a double convex lens, formed of a perfectly transparent fluid called the *crystalline* humour. This lens, in the phenomena of vision, plays the part of the lens of glass O in the camera obscura. The cones of rays coming from visible objects to the eye, having their vertex in this lens, are continued to the posterior surface of the inner chamber of the eyeball, on which is depicted, with its proper form and colours, but in an inverted position, a luminous representation of all the objects of vision; and it is such luminous pictures acting on the optic nerve that produce the effect on the brain which is the immediate cause of vision.

(447.) The whole art of perspective, and therefore a considerable part of the art of the painter, depends upon the properties of conical surfaces. A picture delineated on a plane surface, being intended to produce upon the eye the same effect as visible objects seen at certain distances behind that surface, the relative positions, forms, and magnitudes of the objects on the canvass must be determined by the intersection of the plane of the canvass with the conical surfaces formed by visual rays drawn from the eye of the spectator to the real positions which the objects represented on the canvass are supposed to have. Thus, if we suppose a distant landscape viewed through a rectangular frame placed at a certain distance from the eye of the spectator, a cone, or rather a pyramid, having a rectangular base, must be imagined, the vertex of which shall be at the eye of the

spectator. The frame bounding the landscape, and through which it is viewed, is a section or generatrix of this pyramid; and the diverging faces of the pyramid being continued indefinitely in the direction of the landscape, the actual objects comprehended in it will be included within the four triangular surfaces extending from the eye of the spectator, passing through the four sides of the rectangular frame, and continued indefinitely beyond them. If a line be drawn from the vertex of the pyramid to any point within the limits of the landscape, the place where that line would penetrate the canvass, if canvass were extended in the frame, would be the place of such a point in the painting. If the surface of any object in the view be parallel to the canvass, the section of the cone of which the object is the base made by the canvass will be similar to the object; but if the plane of the object be not parallel to the canvass, then the form of the section of the cone by the canvass will be different from that of the object, and nothing but the application of exact geometrical principles can determine the form of such section. This effect, which, in particular applications of it, is called *fore-shortening*, is one, therefore, which an artist cannot expect to produce with correctness if he be not conversant with the principles of geometry which are required in the solution of such problems. There is, accordingly, no department of the arts of design in which errors so glaring are committed even by the most eminent artists.

The collection of general theorems relating to the intersection of conical and pyramidal surfaces by a plane, which is necessary for the solution of such problems, constitutes the theory of perspective. As an example of such theorems, the following, which are of very universal application and general utility, may be given.

(448.) Parallel lines which are parallel to the plane of the picture will be represented by parallel lines upon the canvass; for if a plane be drawn through any one of these parallels, and through the point of sight

the intersection of such plane with the plane of the canvass will be a line parallel to that through which the plane is drawn, and this line will be that which represents the parallel upon the canvass. Since, therefore, the representations of the parallel lines on the canvass are parallel to the lines themselves, and since the latter are parallel to each other, the lines on the canvass representing them will also be parallel to each other.

(449.) If a system of parallel lines be not parallel to the plane of the drawing, then the lines which represent them on the drawing will be lines which all converge to a point, so placed on the plane of the drawing that a straight line drawn from it to the point of sight will be parallel to the lines thus delineated. For, take any two of the parallels to be delineated, and suppose planes drawn through them, and through the point of sight; these planes will intersect in a certain line parallel to the lines to be delineated, and this line will therefore not be parallel to the plane of the drawing, and will therefore meet it at some determinate point. The intersections of the two planes drawn through the point of sight, and through the two parallels, with the plane of the drawing must meet at the same point, that being in fact the point where all the three planes intersect. That point will therefore be the point to which the representations of the two parallel lines on the canvass must converge, and it may in like manner be shown that all the lines representing the parallels will converge to that point.

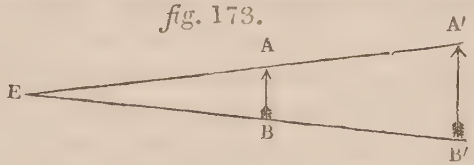
This, in fact, amounts to little more than the statement that all planes which are drawn through a number of parallel lines must have a common line of intersection. For their line of intersection must be parallel to the parallels; and since only one such parallel can pass through the given point, that one must be their common line of intersection.

(450.) These general principles are brought into frequent application in architectural and mechanical drawing, where the forms of the objects represented are

generally determined by systems of parallel lines, as in the case of a building which is composed principally of vertical lines and of horizontal lines at right angles to each other.

(451.) The eye is an organ incapable of estimating actual magnitude. All visible objects appear to the eye of equal magnitudes, provided the angle of the cone formed by the visual rays which bound them is the same. Let E (*fig.* 173.) be the eye, and let A B be an object

fig. 173.



placed at any distance from it, and A′ B′ be another object at a greater distance; if the visual ray from the upper extremity A′ coincide with the visual ray from the upper extremity of the other, and the visual rays from the lower extremities B, B′ also coincide, then the objects will have the same apparent magnitude. In fact the one will entirely cover and intercept the other. In this case, the real magnitudes of the objects will be proportional to their distances from the eye; for they are the bases of similar triangles of which those distances are the sides.

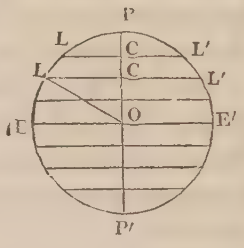
(452.) In general, similar objects will have the same apparent magnitude when their linear dimensions are proportional to their distances from the eye; for in that case their sections are the bases of similar cones of which the altitudes are the distances of the objects from the eye.

(453.) A remarkable example of this is presented by the sun and moon, whose apparent magnitudes are very nearly the same, although the actual diameter of the sun is about 400 times greater than that of the moon. The reason of the equality of their apparent magnitudes is, that while the distance of the moon from the earth is only 240,000 miles, that of the sun is 96,000,000 miles, the one distance being 400 times greater than the other.

CHAP. XVII.

OF SPHERES AND SURFACES OF REVOLUTION.

fig. 174.



(454.) If a circle whose centre is O (*fig.* 174.), be supposed to revolve on a diameter P P′ as an axis, its circumference as it revolves will trace a surface called a *sphere*.

(455.) Since the centre O is equally distant from every point in the revolving circle, and since that circle as it moves coincides successively with every part of the spherical surface, it follows that the point O is equally distant from every part of the surface of the sphere. This point is therefore called the *centre* of the sphere.

(456.) The circle by the revolution of which on its diameter P P′ the spherical surface is produced is called a *meridian* of the sphere.

(457.) All sections of the sphere made by planes passing through P P′ are circles equal to the meridian by the revolution of which the sphere is produced; for the meridian as it revolves coincides successively with all such circles. All such circles are therefore called meridians.

(458.) The diameter P P′ on which the generating circle turns is called the *axis* of the sphere, and its extremities P P′ are called the *poles* of the sphere.

(459.) The axis of the sphere is therefore the common line of intersection of the planes of all the meridians, and the poles are the common points of intersection of the circumferences of such meridians.

(460.) As the meridian revolves, all points, such as L, upon it describe circles whose planes are at right angles

to the axis of the sphere, and whose centres are in the axis at the points where their planes meet the axis.

(461.) The line L L′ perpendicular to P P′ will be the intersection of a plane through L perpendicular to the plane of the generating meridian with the plane of the latter; the line L L′ will therefore be the diameter of the circle described by the point L as the meridian revolves, and C will be the centre of that circle. These circles are sections of the spherical surface made by planes perpendicular to the axis, and are called parallel circles, or simply *parallels*.

(462.) The nearer a parallel is to the centre the greater will be its diameter, and the greatest parallel will therefore be the circle whose diameter is E E′ passing through the centre of the sphere: this circle is called the *equator*.

(463.) The diameter of the equator E E′ being a diameter of the sphere, the equator will be a circle equal to the meridian.

(464.) If the equator itself be taken as a meridian, and one of its diameters as an axis, a sphere would be generated by its motion having the same centre, and the radius equal to that of the original sphere. Every part of the surface of the one sphere being at the same distance from their common centre as every part of the surface of the other sphere, the two spherical surfaces will every where coincide, and they will, in fact, be the same sphere; hence it appears that whatever diameter of the sphere be taken as an axis, the meridians whose planes pass through it will be equal circles, and will by their revolution produce the same spherical surface.

(465.) Hence all sections of a sphere made by planes passing through its centre will be equal circles, whose diameters are equal to that of the sphere: such circles are called *great circles* of the sphere.

(466.) Let a plane intersect the sphere without passing through its centre, and let a diameter of the sphere be conceived to be drawn perpendicular to it; if such diameter be considered as an axis, the plane intersecting the

sphere at right angles to it will form one of a system of parallels, such as L L′, with reference to that axis. The section of the spherical surface by such a plane will be a circle having a diameter, such as L L′, less than the diameter of a sphere: such circles are called *lesser circles* of the sphere.

(467.) Since the radius L C of a lesser circle, the distance of its centre C O from the centre of the sphere, and the radius L O of the sphere form a right-angled triangle, the sum of the squares of L C and C O will always be equal to the square of the radius of the sphere.

(468.) Hence lesser circles whose planes are equidistant from the centre of the sphere are equal.

(469.) The nearer the plane of a lesser circle is to the centre of a sphere, the greater the circle will be.

(470.) If a sphere be rolled in any manner on a plane surface, its centre will move in a plane parallel to that surface, and at a distance from it equal to the radius of the sphere; for the line drawn from the centre to the point where the sphere touches the plane will be the shortest line which can be drawn from the centre of the sphere to the plane, since any other line drawn to the plane must pass beyond the spherical surface before it can meet the plane. The line from the centre of the sphere to the point where the sphere touches the plane is therefore perpendicular to the plane in every position which the sphere can assume: this will, therefore, be the distance between the plane in which the centre of the sphere moves and the plane on which it rolls.

(471.) It is owing to this property that a body of uniform density formed into a perfect sphere will rest indifferently in any position, and roll indifferently in any direction on a horizontal plane; for its centre of gravity, coinciding as it must with its centre of magnitude, moves in a horizontal plane; and as it never, therefore, has a tendency either to ascend or descend, the body will indifferently rest or move in any direction in virtue of a well-known property of the centre of gravity.

(472.) The sphere is unique in the possession of this property, and all the effects produced by the skill of the billiard-player are connected with it. The billiard-table is, or ought to be, an exact horizontal plane surface, and the billiard-ball should be a sphere of uniform density, a property which ivory possesses in a very high degree. The centre of the ball throughout all its motions on the table is therefore at the same absolute height above the surface of the earth; and its motions consequently, being free from any effect of gravity, are governed exclusively by the impulses given to it by the player.

(473.) The earth has very nearly the form of a sphere; the highest mountain and the lowest depths of the sea do not amount to $\frac{1}{1000}$ part of its diameter, and form relatively to its magnitude inequalities much less considerable than the roughness on the rind of an orange. There is another slight departure from the exact spherical form which gives to the earth a figure slightly approaching that of a turnip; but this is so extremely minute in degree, that a billiard-ball having the same want of perfect sphericity would not be known by mere inspection to be imperfect in its form.

(474.) Considering the earth then as a sphere, it has a motion of rotation on one of its diameters precisely similar to that by which we have shown that a spherical surface is produced; this diameter is called the *axis of the earth*, and its extremities are called the *poles*. The points of the earth's surface as they revolve move in planes at right angles to the axis.

(475.) The sections of the earth at right angles to the axis are called *parallels of latitude*; and, according to what has been already proved, these parallels are less as they approach the poles.

(476.) The great circle at right angles to the axis is the *equator*, which divides the globe into the northern and southern hemispheres. The great circles whose common intersections are the poles are called *terrestrial meridians.*

(477.) The distance of any place from the equator measured upon a meridian passing through that place, and expressed in degrees, minutes, and seconds, is called the *latitude* of the place.

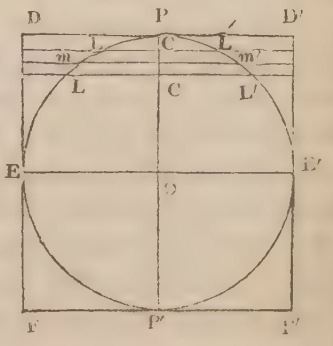
(478.) All parts of the same parallel of latitude, being at the same distance from the equator, have the same latitude.

(479.) If three hundred and sixty meridians be drawn whose planes shall divide the space around the axis of the earth into three hundred and sixty equal angles, these meridians will divide the equator and every parallel of latitude into three hundred and sixty equal parts or degrees: these are called degrees of *longitude;* and the difference of the longitudes of any two places on the earth will accordingly be measured by the angle formed by the planes of the meridians which pass through them, or, what is the same, it will be measured by the arc of the equator intercepted between such meridians.

(480.) Different nations have adopted different points of departure from which the longitudes of places are measured. The English measure all longitudes from the meridian which passes through the Observatory at Greenwich, and the French adopt as their zero of longitude the meridian which passes through the Observatory at Paris.

(481.) If the surface of a sphere be divided into a number of parallel bands by the planes L L′ (*fig.* 175.) of parallel circles, the surfaces of these bands may be considered as equivalent to those of truncated cones, when the planes of the circles L L′ are so near each other that the curvature of the spherical surface L L between them is inconsiderable. It is evident that if L L be considered as a straight line, the revolution of

fig. 175.

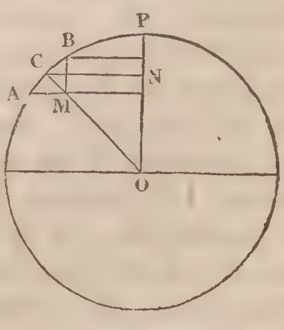


the figure round the axis C C would produce a truncated cone whose bases are the circles L L′. The conical surface included between these bases may then be regarded as a part of the surface of the sphere.

(482.) The area of the surface of a truncated cone being equal to its side multiplied by half the sum of its bases, it follows, that when the parallels L L′ (*fig.* 175.) are very close together, the area of the spherical surface, included between them, will be equal to the distance L L between the parallels multiplied by half the sum of the circumferences of the two parallels, or, what is the same, by the circumference of a parallel $m m'$ taken midway between them.

(483.) Round the circle (*fig.* 175.), let a square D F′ be circumscribed. By the revolution of the figure on the axis P P′, as the circle describes a sphere, the square will describe a cylinder circumscribing that sphere, and the planes of the parallels will intercept between them a cylindrical surface, which shall be equal to the part of the spherical surface intercepted between the same planes. For, by what has been already proved, the cylindrical surface intercepted between these planes is equal to the rectangle under the distance C C between the planes and the circumference of a circle whose diameter is E E′, while the spherical surface has just been proved to be equal to the rectangle under the line L L and the circle whose diameter is $m m'$; but we shall now prove that these rectangles are equal; and hence it will follow, that two parallel planes at right angles to the axis P P′, when very close together, will intercept equal magnitudes of the surface of the sphere and the circumscribed cylinder. To prove the equality of the abovementioned rectangles, let A B (*fig.* 176.) represent the arc L L (*fig.* 175.), the arc A B being considered so small that

fig. 176.



it may be regarded as a straight line; let C be its middle point, and let CN be drawn perpendicular to OP; draw BM perpendicular to CN: the triangle AMB will then be similar to the triangle CNO, the sides of each being perpendicular to those of the other. We shall have, therefore, the following proportion:—

$$CN : CO = BM : BA$$
$$\text{or } 2\,CN : 2\,CO = BM : BA.$$

Hence, the rectangle under BA and twice CN, or, what is the same thing, the rectangle under LL and *mm′*, will be equal to the rectangle under BM and twice CO, or, what is the same, under CC and EE′; but, since these rectangles are equal, the rectangle under LL and the circumference of the circle whose diameter is *mm′* is equal to the rectangle under CC and the circumference of the circle whose diameter is EE′, but these are equal, respectively, to the truncated conical surface between the planes LL′ and the cylindrical surface between the same planes.

(484.) Since the portions of the cylindrical and spherical surfaces intercepted between parallel planes drawn very close together are equal, the portion of such surfaces between parallel planes at any distances whatever are equal; for such portions will be made up of a number of narrow bands intercepted by parallel planes very close together.

(485.) Hence, the surface of the entire sphere is equal to the surface of the entire cylinder.

(486.) Since the surface of the cylinder is equal to the rectangle under the circumference of its base and its height (389.), and since its base is equal to a great circle of the sphere, and its height is equal to a diameter of the sphere, it follows, that the surface of the cylinder is equal to the rectangle under the circumference of a great circle of the sphere and its diameter.

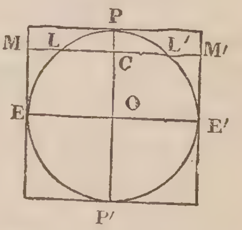
(487.) Since the area of a great circle is equal to half the rectangle under its circumference and radius (223.), four times the area will be equal to the rectangle

under the circumference and diameter. Hence it follows, that the cylindrical surface circumscribed round the sphere will be equal to four times the area of a great circle.

(488.) Since this cylindrical surface is equal to the area of the surface of the sphere, it follows, that the area of the surface of a sphere is equal to four times the area of one of its great circles.

(489.) The area of the surface of a spherical segment L P L′ (*fig.* 177.), will be equal to the area of the cylindrical surface, the diameter of whose base is M M′ and whose height is P C.

fig. 177.



(490.) Such cylindrical surface is equal to the rectangle under the circumference of a circle whose diameter is M M′ or E E′ and P C. Hence the surface of the spherical segment is equal to the height P C of the segment multiplied by the circumference of a great circle.

(491.) Hence the surfaces of segments of the same sphere are proportional to their heights, and those of different spheres are proportional to the rectangles under their heights and the diameters of the spheres.

(492.) These properties supply the means of calculating the quantity of matter necessary to coat, cover, or line a sphere or any part of a sphere; thus, it is evident, that the quantity of paint necessary for a sphere is four times the quantity which would be sufficient for the surface of a great circle of the same sphere.

The quantity of lead, copper, or zinc, necessary to cover a hemispherical dome, would be twice the quantity which would cover the base of that dome, and so on.

(493.) If the surface of a sphere be conceived to be made up of an infinite number of small polygons with plane faces, the volume of the sphere will, like that of other solids, be resolved into a corresponding number of pyramids having the centre of the sphere for their common vertex; the volume of the sphere will therefore

be equal to that of a single pyramid whose base shall be equal to the sum of the bases of all the component pyramids—that is, to the surface of the sphere, and whose altitude is equal to their common altitude — that is, to the radius of the sphere. The volume of the sphere is therefore equal to the volume of a pyramid or cone, whose base is equal to the surface of the sphere, and whose altitude is equal to its radius.

(494.) Hence the volume of a sphere is equal to four times the volume of a cone whose base is a great circle of the sphere, and whose altitude is the radius of the sphere, or to twice the volume of a cone with the same base, and whose altitude is the diameter of the sphere.

(495.) The volume of a sphere is also equal to the volume of a cylinder whose base is equal to the surface of the sphere, and whose altitude is equal to one third of the radius of the sphere; for such cylinder is equal to a cone or pyramid whose base is equal to the surface of the sphere, and whose altitude is equal to its radius.

(496.) The volume of the sphere is therefore equal to four times the volume of the cylinder whose base is a great circle of the sphere, and whose altitude is one third of the radius of the sphere; and therefore the volume of a sphere will bear to the volume of a cylinder whose base is a great circle, and whose altitude is the diameter of the sphere, a ratio of 4 to 6, or of 2 to 3.

(497.) The volume of a sphere is therefore two thirds of the volume of a circumscribed cylinder.

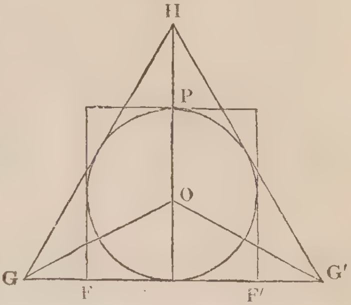
(498.) Since the surface of the circumscribed cylinder is four times the area of a great circle, and its ends are equal to great circles, the whole surface of the cylinder, including its ends, is equal to six times the area of a great circle, it appears therefore that the surface of a sphere is two thirds of the entire surface of the circumscribed cylinder.

(499.) The surface and volumes of the sphere and

circumscribed cylinder are therefore both in the ratio of two to three.

(500.) If a square and equilateral triangle be circumscribed round the same circle (*fig.* 178.), and all the figures revolve together round the axis H P′, a sphere, circumscribed cylinder, and circumscribed equilateral cone will be formed, and the volumes as well as the entire surfaces of these three solids will be in the continued ratio of 2 to 3. This has been already proved with respect to the sphere and cylinder, and we shall now show that the proportion of the surface and volume of the cylinder to those of the cone will be in the ratio of 2 to 3.

fig. 178.



It has been shown that the volume of the cylinder is equal to the area of a great circle of the sphere multiplied by twice its radius. We shall now show that the volume of the cone is equal to three times the area of such a circle multiplied by twice the radius. The area of the base of the cone will be to that of the base of the cylinder, as the squares of their diameters — that is, as the square of G G′ is to the square of F F′; but the square of G G′ is four times the square of G P′, or four times the difference between the squares of O G and O P′, or, what is the same, to four times the difference between the squares of O H and O P′; but we shall prove that O H is double O P′, and therefore the difference between the squares of O H and O P′ is three times the square of O P′. If the line O G′ be drawn, it is evident that the triangles into which the equilateral triangle is divided by the lines O G, O G′, and O H, are equal, and therefore the area of each is one third of the area of the whole. The altitude therefore of the triangle G O G′ is one third of the altitude of the triangle G H G′, that is, O P′ is one third of H P′, and therefore H O is double O P′.

Since, then, the square of G G′ is four times the difference between the squares of H O and O P′, it is twelve times the square of O P′, and therefore three times the square of P P′ or of F F′. The square of the diameter of the base of the cone is therefore three times the square of the diameter of a great circle, and the area of the base of the cone is therefore three times the area of a great circle; but the altitude of the cone H P′ is equal to three times the radius O P′ of a great circle; therefore the volume of the cone will be equal to three times the area of a great circle multiplied by the radius, while the volume of the circumscribed cylinder is equal to the area of a great circle multiplied by its diameter, or to twice the area multiplied by its radius; the volume of the cone will therefore be to the volume of the cylinder in the ratio of 3 to 2.

The surface of the cone, exclusive of its base, will be equal to half the rectangle under its side G H and the circumference of its base; but since G H is equal to G G′, this will be equal to half the rectangle under the diameter of its base and the circumference of its base; therefore the conical surface will be equal to the rectangle under the radius and circumference of its base, or to twice the area of its base; and therefore the whole surface of the cone, including its base, is equal to three times the area of its base, or to nine times the area of a great circle of the sphere; but the entire surface of the cylinder, including its ends, has been proved to be equal to six times the area of a great circle, and therefore these areas are in the ratio of 9 to 6, or of 3 to 2.

(501.) If the firmament be viewed with attention on a cloudless night, from the deck of a ship, with no land in view, the spectacle which will be presented to the eye will be that of an enormous hemispherical surface glittering with stars, and having the sea for its circular base. So far as the eye can inform us, all the objects visible in the heavens are equally distant, and the

boundary of the view in the horizontal direction is a circle formed by the intersection of the plane of the water with the hemispherical celestial vault. The stars, which are so richly and abundantly scattered over the firmament, will appear to maintain, with respect to each other, the same relative position as if each was fastened immoveably in the surface of the heavens. If, however, the firmament be attentively watched for some hours, its entire position with respect to the base of the hemisphere will appear to be changed, not by any disturbance of the arrangement or relative position of the bodies upon it, but by a general shifting of the position of the whole vault,—the stars being carried with it. If, during these changes, a line be extended from the eye of the spectator to any individual star, and be kept in the direction of that star, the position of this line will be observed to change as it follows the motion which the star has in common with the firmament; and if the course of the line thus moving be observed, it will be found to move in the surface of a cone of which the eye of the spectator, or, what is the same, the centre of the hemisphere, is the vertex. If such a line be conceived to be continued to the star, the base of this cone would evidently be a circle described by the motion of the star on the celestial sphere.

The motion of all the stars being observed in this way, it is found that they move in parallel circles on the sphere, and with such motions as are consistent with the preservation of their relative position. In fact, their motions are such as would be produced, if the whole celestial sphere revolved on a diameter as an axis passing through a certain point in the heavens, which alone appears to be at rest. This point is called the *celestial pole.*

This apparent motion of the heavens, which was long supposed to be produced by a real motion of the universe daily round the earth, is now known to be merely the effect of the diurnal rotation of the earth

upon its axis, the effect of which is to give to all visible objects round the earth an apparent motion in a contrary direction, just as the banks of a river, viewed from the cabin of a boat, appear to move in a direction contrary to the boat itself.

(502.) The sphere has a remarkable and important property analogous to one already mentioned as belonging to a circle, but which does not admit of demonstration on any principles of reasoning sufficiently simple and elementary to be introduced here. In virtue of this property, a sphere is the solid figure which, within a given surface, contains a greater volume than any other solid figure, or, what amounts to the same, a given volume has the least surface when it takes the figure of a sphere.

(503.) The nearer the form of any solid approaches to that of a sphere, the greater volume it will contain within a given surface.

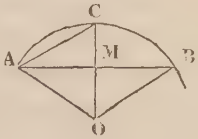
(504.) The mutual attraction which the particles of matter have for one another, always gives them a tendency, when their motion is unobstructed, to collect themselves within the smallest possible superficial dimensions.

When vapour is condensed in the clouds and converted into liquid by cold or other physical agency, its molecules, attracting each other, form into spherules, and descend in drops of rain. If quicksilver be let fall upon any surface which has no attraction for it, the mutual attraction of the particles of the liquid will cause it to collect in globules. These are only manifestations of the tendency of matter, by the reciprocal attraction of its particles, to collect within the smallest possible dimensions, and are practical demonstrations that a sphere contains a greater volume than another solid of the same surface. The spherical form affected by the great bodies of the universe,—the sun, planets and satellites forming our own system, besides those which compose the numberless systems

which the power of the telescope has disclosed to us,— are examples of the same principle on a greater scale.

(505.) If O (*fig.* 179.) be the centre of a circular sector O A B, and O C be the radius bisecting its angle, and A B be the chord of its arc, this figure, by revolving round O C as an axis, will generate the *sector of a sphere.*

fig. 179.



(506.) As the sector of a circle consists of a triangle and segment, the sector of a sphere consists of a cone and a spherical segment. The chord A B, as the sector revolves round O C, produces a circle, the area of which is the common base of the spherical segment, and the cone of which the spherical sector is formed.

(507.) The volume of a spherical sector is found by multiplying the area of its spherical surface by one third of its radius, being equal to the volume of a cone whose area is that surface, and whose altitude is the radius. This is demonstrated by the method already applied to the determination of the volume of a sphere.

(508.) It has been already proved that the surface of a spherical segment A C B is equal to the rectangle under C M, the altitude of the segment, and the circumference of a great circle. The circumference of such a circle is to the circumference of a circle whose radius is the chord A C, as the diameter of the sphere is to twice the chord A C, or, what is the same, as the chord A C itself is to twice C M. The rectangle, therefore, under half the chord A C and the circumference of a circle of which it is the radius, will be equal to the rectangle under C M, and the circumference of a great circle. Hence it follows, that the area of the surface of the spherical segment A C B is equal to the area of a circle whose radius is A C.

(509.) The volume of a spherical sector is therefore found by multiplying the area of a circle whose radius is the chord of half the generating arc of the sector by one third of the radius.

(510.) The volume of the cone whose base is A B being substracted from the volume of the sector, the volume of the spherical segment will remain; but the volume of the cone is equal to one third of its altitude M O multiplied by the area of the circle whose radius is A M, and the volume of the sector is equal to one third of the radius A O multiplied by the area of the circle whose radius is A C; the difference between these products will therefore be the volume of the spherical segment.

(511.) It has been shown that the surfaces of cylinders and cones are of such a nature, that if any thin covering attached to them were separated from them or unrolled, it would admit of being spread out upon a plane without wrinkling or being torn; the surface of a sphere, however, does not possess this quality. If a thin skin or covering attached to a sphere were removed from it and laid upon a plane, it could not be brought in contact with the plane in every part. Any attempt to produce such an effect would either tear the substance, or produce wrinkles or folds in it.

(512.) Surfaces which, like those of cylinders and cones, admit of having a plane cloth rolled upon them, so as to cover them in every part without wrinkles or tears, are distinguished, by being called *developable surfaces*, from others which, like the surface of a sphere, do not possess this property.

(513.) This circumstance produces a difficulty in lining or coating spherical surfaces in the arts with cloth, or in plating them with metal, which does not exist in the case of cylindrical or conical surfaces. If it be required to cover or line an archway with cloth or plates of metal, the lining may be laid on in pieces of any magnitude — being easily curved so as to adapt itself to the shape of the arch; but if it be required to line or cover a hemispherical dome, this cannot be done, and expedients must be adopted to divide the lining or covering material into pieces of such magnitude and form

as, when placed in juxtaposition, will as nearly as possible cover the spherical surface.

(514.) Two methods are resorted to for accomplishing this. Let the spherical surface be divided by a number of parallel circles A B, A′ B′ (*fig.* 180.) into parallel zones of very small breadth, so that the arcs of a meridian A A′ or B B′, intercepted between them, may be regarded as straight lines. The surface of such a zone may then be considered as that of a truncated cone whose bases are the parallel circles A B, A′ B′. Let Z′ (*fig.* 181.) be taken as a centre, and Z′ *a* as a radius

fig. 180.

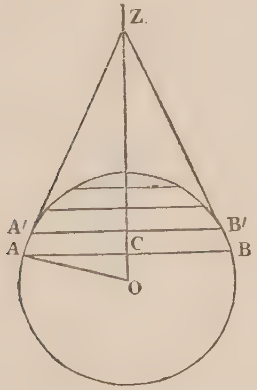
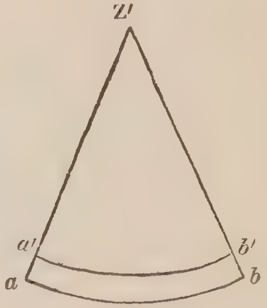


fig. 181.



equal to Z A (*fig.* 180.), and let a circular arc *a b* be described equal in length to the circumference of the parallel A B, and taking Z′ *a*′ (*fig.* 181.) equal to Z A′ (*fig.* 180.) and describing the arc *a*′ *b*′, it will be equal to the circumference of the parallel A′ B′. In fact, if the surface of the zone be unrolled from the sphere and spread out, it will form the band *a a*′ *b*′ *b* (*fig.* 181.), bounded by the two parallel arcs.

If, therefore, the radius Z A be known, and the circumference of the parallel A B at any point of the sphere, a narrow zone may be formed of any substance, which shall surround the sphere at that place, and be every where in contact with it; such a zone will be obtained by describing on a plane surface the sector of a

circle whose radius is equal to Z A, and whose arc is equal to the circumference of the circle A B.

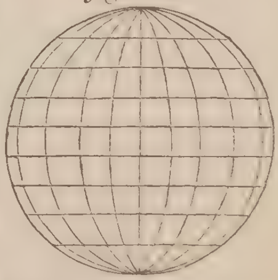
The length of Z A, corresponding to any given point on the sphere, is easily obtained. The diameter A B being known, the radius A C is known, the square of which being taken from the square of A O the radius of the sphere, the remainder will give the square of C O, which will therefore be known; but the ratio of C O to C A will be the same as that of O A to A Z. The length of A Z will therefore be determined.

The angle Z′ (*fig.* 181.), which, with a radius equal to Z A (*fig.* 180.), will give an arc *a b* equal to the circumference of the parallel A B, may be easily determined; for this angle, expressed in degrees, will bear to 360 degrees the same proportion as the circumference of the circle A B bears to the circumference of a circle whose diameter is twice Z A. The angle Z′ (*fig.* 181.) will therefore be found by multiplying 360 degrees by C A, and dividing the product by Z A.

In this manner a series of narrow zones may be formed, which, when laid upon the sphere, will very nearly cover it,— the edges uniting without perceptible folds or wrinkles; and the more narrow such zones are formed the more nearly will they cover the sphere.

(515.) Another method of covering a spherical surface consists in dividing it by a number of meridians, as represented in *fig.* 182., forming with each other angles so small that the arcs of parallel circles intercepted between them may be considered as straight lines. If these meridians be themselves divided into small and equal arcs by parallel circles intersecting the axis of the sphere at right angles, the whole spherical surface will be divided into small quadrilateral figures bounded by the parts of the meridians and parallels, which may be considered as plane trapeziums: the form and magnitude of each series of these

fig. 182.



being determined, and the substance intended to cover the sphere being resolved into corresponding pieces, the object of covering the sphere by plane figures will be attained; and the precision with which this will be accomplished will be proportional to the smallness of the pieces into which the sphere is divided.

(516.) The sphere is not the only surface which can be formed by the revolution of a circle round a straight line: we have seen that a semicircle revolving on its diameter will generate a sphere; but other segments revolving on their chords will generate solids of other forms. Thus a segment less than a semicircle revolving on its chord P P′ (*fig.* 183.) will generate a solid, such as there represented, having pointed ends.

fig. 183.

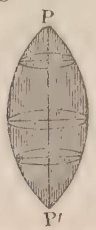
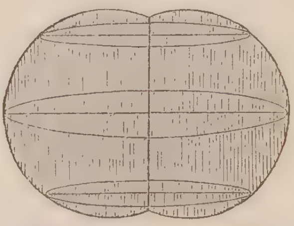
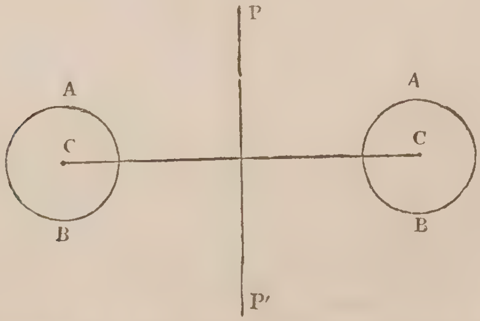


fig. 184.



A segment of a circle greater than a semicircle, as represented in *fig.* 184., revolving on its chord as an axis, will generate a figure such as represented in *fig.* 184., having a hollow at top and bottom resembling that of the end of an apple from which the stalk proceeds.

fig. 185.



If a circle A B (*fig.* 185.) revolve round a line such as P P′, drawn in its plane, but outside it as an axis, it

will generate an *annulus*, the centre C of the circle describing a circle round P P′, which will be the axis of the annulus.

Such a solid is represented in perspective in *fig.* 186.

If an arc of a circle such as A B revolve round a line P P′ drawn on the convex side of it and in its plane, as an axis, it will generate a figure with concave cylindrical sides, such as is represented in *fig.* 187.

fig. 186.

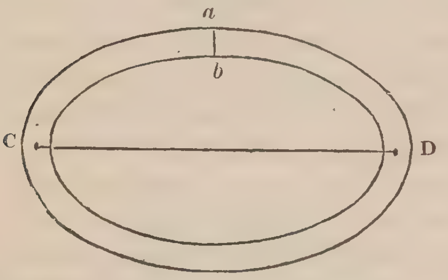
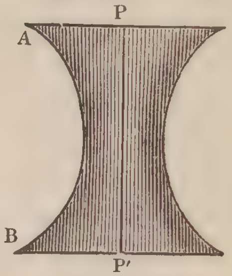
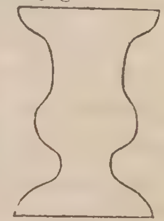


fig. 187.



(517.) Almost all the variety of vases of metal and porcelain used in domestic economy, ancient and modern, and adopted for ornamental purposes in the arts, are surfaces produced by the revolution of the arcs of curves round lines drawn in their planes, within or without them, in the manner above described, combined with cylindrical surfaces, and those of truncated cones; all the surfaces of revolution composing the same vessel having a common axis, as represented in *fig.* 188.

fig. 188.



(518.) A circle is not the only line by the revolution of which round a fixed axis a surface may be generated; on the contrary, this method of producing a surface is general, and has given rise to a class of surfaces called *surfaces of revolution*, and which are the class of geometrical forms of the most frequent occurrence both in natural and artificial productions. Any line whatever, whether straight or curved, may revolve round another line as an axis, and by such revolution it will generate a surface of revolution, the form and properties of which will depend on the species of line which

revolves, and its position with respect to the axis of revolution.

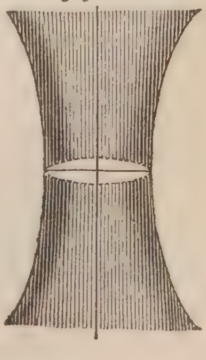
The right circular cylinder and cone, as has been already observed, belong to the family of surfaces of revolution. If one of two parallel right lines revolve round the other as an axis, it will produce the surface of a right circular cylinder; and if one side of a plane rectilinear angle revolve round its other side as an axis, it will produce the surface of a right circular cone.

(519.) From the mode in which they are generated, it follows, that the sections of all surfaces of revolution made by planes at right angles to the axis of revolution, are circles having their centres in the axis of revolution: this is a characteristic property of such surfaces; and, as it belongs to none other whatever, it may be, and sometimes is, taken as the basis of their definition. It is evident, that, in the production of a surface of revolution, all the points of the revolving line move in parallel planes, and, as they preserve their distances from the axis of revolution, each must describe a circle whose centre is in that axis.

(520.) Surfaces of revolution are infinitely various, not only in consequence of the great variety of lines by the revolution of which they may be produced, but by reason of the variety of surfaces which may be produced by the same line revolving under different circumstances.

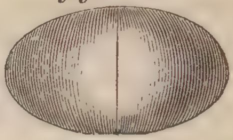
When a right line is in the same plane with the axis round which it revolves, it will produce, as has been shown, either a cylindrical or conical surface, according as it is parallel or not to the axis of revolution; but if it be not in the same plane with the axis of revolution, it will produce a curved surface (*fig.* 189.) whose cross section shall be a circle whose radius is the least distance of the revolving line from the axis of revolution.

fig. 189.



(521.) Among the productions of nature, the great bodies of the universe — the sun, planets, and satellites — are surfaces produced by the revolution of an oval or ellipse round its lesser axis (*fig.* 190.).

fig. 190.



Fruit of almost every kind are surfaces of revolution produced by the segment of a circle revolving round a chord.

A lemon affords an example of a surface of revolution (*fig.* 191.) formed by a segment less than a semicircle revolving on its chord.

fig. 191.

An apple (*fig.* 184.), of a surface formed by a segment greater than a semicircle revolving on its chord.

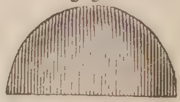
An orange is an example of a surface of revolution (*fig.* 190.) formed by an oval revolving on its shorter axis.

A plum (*fig.* 192.), of a surface of revolution formed by an oval revolving on its longer axis.

fig. 192.

(522.) Every species of dome in architecture is a surface of revolution. A hemispherical dome is formed by a semicircle revolving round the radius which is perpendicular to its diameter (*fig.* 193.).

fig. 193.



An oblate elliptical dome (*fig.* 194.) is a surface of revolution produced by the revolution of a semi-ellipse round its lesser semi-axis.

fig. 194.

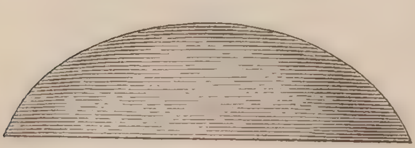
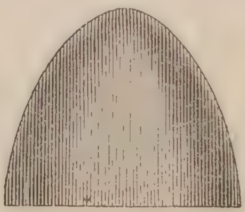


fig. 195.



A prolate elliptical dome (*fig.* 195.) is produced by the revolution of a semi-ellipse round its greater semi-axis.

(523.) The art of turning consists chiefly in the production of surfaces of revolution. The cutting tool imparts the circular form to the body, which is turned by the lathe, and if the cutting tool itself be guided along the lines, by the revolution of which the surface is supposed to be formed, the requisite form will be imparted to the body submitted to the operation. Thus if the cutter be moved along a line parallel to the axis of the lathe, a cylinder will be formed; if it be moved along a straight line, intersecting the axis of the lathe, a cone will be formed; if it be moved along a straight line which is not in the plane of the axis of the lathe, a surface will be formed like that represented in (*fig.* 189.); if it be moved in a semicircle, whose centre lies in the axis of the lathe, a sphere will be formed; and if it be moved in a semi-ellipse, a spheroid will be formed, and so on.

CHAP. XVIII.

OF THE REGULAR SOLIDS.

(524.) A REGULAR solid is a solid all the faces of which are regular polygons, or, rather, regular plane figures; that is, figures which are equiangular and equilateral.

(525.) It is easy to prove that there cannot be more than five regular solids.

1. If the faces be equilateral triangles, solid angles may be formed by their combination in different ways. Three, four, or five angles of 60° may form a solid angle; but if six or more such plane angles were united edge to edge, they would be equal to or greater than 360°, and consequently could not form a solid angle, since the sum of the plane angles forming a solid angle must evidently be less than 360°.

The number of regular solids, therefore, whose faces are equilateral triangles cannot exceed three.

2. If the faces be squares, a solid angle can be formed by three right angles, but not by four, or any greater number, since the sum of four right angles is equal to 360 degrees. There cannot, therefore, be more than one regular solid with square faces.

3. Suppose the faces are regular pentagons. A solid angle may be formed of three angles of a regular pentagon, for the magnitude of the angle of a regular pentagon is six-fifths of a right angle, and therefore the aggregate magnitude of three such angles is eighteen-fifths of a right angle, or three right angles and three-fifths, which being less than four right angles, a solid angle may therefore be formed by three angles of a regular pentagon. But four or more such angles, being

greater than four right angles, cannot form a solid angle. Hence there cannot be more than one regular solid with pentagonal faces.

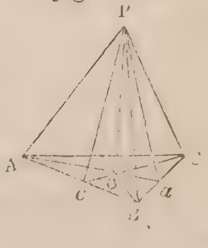
4. Suppose the faces were regular hexagons. The angles of a regular hexagon are 120 degrees, and three such angles would therefore be equal to 360 degrees. Three angles of a regular hexagon combined would therefore form a plane, and could not form a solid angle; and as four or more such angles would be greater than 360 degrees, they could not form a solid angle.

5. The angles of all regular polygons having more than six sides are greater than one third of four right angles. Consequently three or more such angles combined, amounting to more than 360 degrees, cannot form a solid angle.

Hence no regular solid can have faces with more than five sides. Hence we infer, *first*, that there cannot be more than five regular solids; *secondly*, that of these, three have triangular faces, one has a square face, and one a pentagonal face; *thirdly*, that the solid angles of the three regular solids having triangular faces are formed of three, four, and five plane angles, and that the solid angles of the others are formed of three plane angles.

(526.) To construct a regular solid having triangular faces, whose solid angles shall be composed of three plane angles, let A B C (*fig.* 196.) be one of the sides of such a solid, and let O be the centre of this equilateral triangle, taken upon the perpendicular from the angle A to the side B C, at a distance O *a* from that side equal to one third of the length of A *a*. From the point O draw a perpendicular O P to the plane of the triangle A B C. From the three angles A, B, C, let lines be inflected on this perpendicular equal to the sides of the equilateral triangle A B C. These lines will meet the perpendicular at the same point P, and

fig. 196



will form the edges of a triangular pyramid, whose faces will be equilateral triangles equal to the base A B C.

To prove that the lines thus inflected will meet the perpendicular O P at the same point, let A P be one of those lines. In the right-angled triangle A O P the square of O P will be equal to the difference between the squares of A P and A O; but the point O being at equal distances from each of the three angles of the triangle A B C, the height of the point at which each of the inflected lines will meet the perpendicular above the point O will be the same, its square being equal to the difference of the squares of the equal inflected lines and the equal distances of the point O from the three angles.

The inclinations of the planes of every pair of faces are equal. Since A a and P a are both drawn to the middle point of the common base of the equilateral triangles B A C and B P C, they will be perpendicular to that base, consequently, the angle P a A will be the angle under the planes of the two triangles. For the same reason, P c C will be the angle under the planes of the faces A P B and A C B. But since the sides of the triangle A a P, are equal respectively to the sides of the triangle P c C, the angles of these triangles are equal, and therefore the faces P A B and P B C of the pyramid are equally inclined to the plane of its base.

In the same manner, it may be shown that the planes of all the faces of the solid are equally inclined to each other.

(527.) This regular solid, with four equal and similar triangular faces, is called the *regular tetraedron.*

(528.) To determine numerically the volume of a regular tetraedron, whose side is the linear unit. Since A B is the unit, B a will be $\frac{1}{2}$, and therefore the square of B a will be $\frac{1}{4}$; but the square of A a is the difference between the squares of A B and B a, and is therefore $\frac{3}{4}$. But A O being $\frac{2}{3}$ of A a, its square will be $\frac{4}{9}$ of the square of A a, therefore the square of A O is $\frac{4}{9}$ of $\frac{3}{4}$, or

$\frac{3}{9}$, or $\frac{1}{3}$. But since A O P is a right angle, the square of O P is the difference between the squares of A P and A O, that is the difference between 1 and $\frac{1}{3}$, or $\frac{2}{3}$. Since then the square of P O is $\frac{2}{3}$, the line P O itself will be $\sqrt{\frac{2}{3}}$.

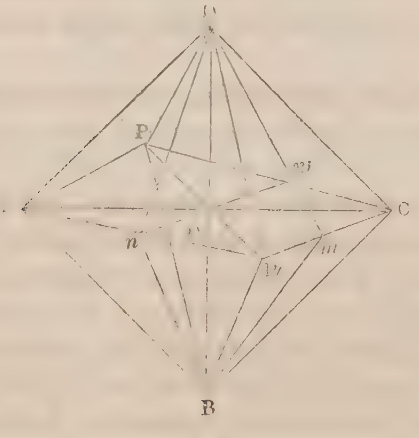
But since the square of A a is $\frac{3}{4}$, the line A a itself is $\frac{\sqrt{3}}{2}$, and this being multiplied into half of B C, which is $\frac{1}{2}$, will give $\frac{\sqrt{3}}{4}$ for the area of the triangle A B C. This area being multiplied by $\frac{1}{3}$ of the perpendicular P O, will give the volume of the pyramid. This volume is therefore $\frac{\sqrt{3}}{4} \times \frac{1}{3} \sqrt{\frac{2}{3}} = \frac{1}{6\sqrt{2}}$.

(529.) The volume of a tetraedron is to that of a cube with an equal edge, therefore as 1 is to six times the square root of 2, or as 1 is to 8·485.

(530.) To construct a regular solid with triangular faces, and whose solid angles are formed by four plane angles.

Construct a square A B C D (*fig.* 197.), and through its

fig. 197.



centre O, draw a perpendicular to this plane, extending it both above and below the plane; from the points A, B, C, D, inflect on this perpendicular, at both sides of the plane, lines equal to the sides of the square. It is evident that those four which lie on the same side of the plane of the square will meet the perpendicular at the same point; let these two points be P and P′: two pyramids will thus be constructed on opposite sides of the square, the faces of which will be the equilateral triangles, whose bases are the sides of the square. These two pyramids, having the square as their common base, will form a regular solid with eight triangular sides, of which the square is a diagonal plane.

(531.) This solid, with eight equilateral triangular faces, is called the *regular octaedron*.

(532.) The inclinations of the planes of every pair of adjacent faces of the solid are equal.

From D and B draw lines to the middle points m, n of the edges P′ C and P′ A. These lines will be perpendicular to P′ C and P′ A, and therefore contain angles D m B and D n B, equal to the inclinations of the planes D P′ C, B P′ C, and D P′ A, B P′ A. But they are equal, being the altitudes of equal equilateral triangles, and therefore the isosceles triangles D m B and D n B having the common base D B, are equal, and the angles D m B and D n B, which determine the inclinations of the planes, are equal; and in the same manner, the inclinations of other pairs of adjacent faces may be proved to be equal.

(533.) Hence, the inclination of the faces is equal to the vertical angle of an isosceles triangle, whose base DB is to its side D n as the hypothenuse of a right angle isoceles triangle is to the altitude of an equilateral triangle constructed on one of its sides.

(534.) If three faces of the octaedron, whose bases form the edges of the same face, such as ADP, BCP, AP′B, be continued through those sides until they form a solid angle, they will form a regular tetraedron with the face through whose sides they are produced.

(535.) Each pair of faces of the octaedron, such as A P B and D P′ C, which are constructed on opposite sides A B, D C of the square, and also on opposite sides of its plane, are parallel; for the alternate angles which their planes form with that of the square, are equal.

(536.) If the planes of three faces, which are terminated in the edges of any one face A B P, be produced until they form a solid angle, and also until they meet the plane of the face D C P′, which is parallel to A B P produced, they will with it form a regular tetraedron circumscribing the octaedron. Each face of this tetraedron will be divided into four equilateral triangles by the edges of the face of the octaedron by whose production it is formed. Hence it follows, that the whole surface of this tetraedron is sixteen times one of the faces of the octaedron, and is, therefore, double the whole surface of the octaedron.

(537.) It appears, therefore, that if the four corners be cut from a regular tetraedron by planes through the points of bisection of every three adjacent edges, the remaining figure will be a regular octaedron. Since each pyramid thus cut off, is similar to the whole, and the edges are in the proportion of one to two, the volume of each pyramid cut off will be $\frac{1}{8}$ of the whole; therefore the volume of each of the four pyramids removed, will be $\frac{1}{4}$ of the volume of the remaining octaedron.

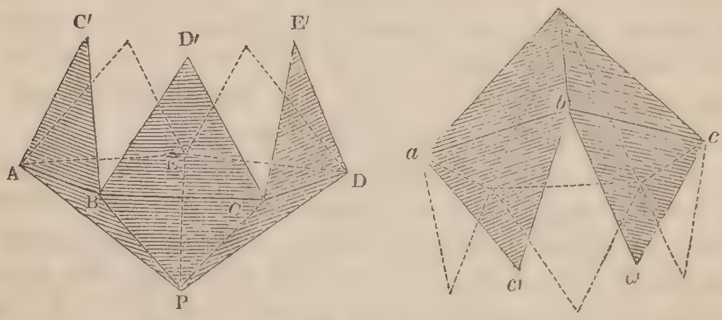
(538.) Hence it appears, that the volume of a regular octaedron, whose edge is the unit, will be half the volume of a regular tetraedron whose edge is 2. But by (528.), the volume of a tetraedron, whose edge is 1, is $\frac{1}{6\sqrt{2}}$; and since a similar solid, whose edge is 2, has 8 times the volume (364.), it follows, that the volume of a tetraedron, whose edge is 2, is $\frac{8}{6\sqrt{2}} = \frac{2\sqrt{2}}{3}$.

Hence the regular octaedron, whose edge is 1, is $\frac{\sqrt{2}}{3}$.

The volumes of an octaedron and cube having the same edge will therefore be in the proportion of the square root of 2 to 3, or as 1414 to 3000 very nearly.

(539.) To construct a regular solid with triangular faces, and whose solid angles are formed by five plane angles. Let a regular pentagon A B C D E (*fig.* 198.) be constructed, and through its centre let a perpendicular to its plane be drawn. From the vertices of its five angles let right lines, equal to its sides, be inflected on this perpendicular. Since the side of a regular pentagon is greater than the radius of its circumscribing circle, these lines will meet the perpendicular below the plane of the pentagon; and since the lines so inflected are equal, they

fig. 198.



will meet the perpendicular at the same point P, so as to form a regular pentagonal pyramid. The solid angle P at the vertex of this pyramid will then be formed by five plane angles, each of which is 60°. Two of the plane angles which form each solid angle at the base of the pyramid have evidently the same inclination as any two of the plane angles which form the solid angle P, being, in fact, the same planes. Hence, the solid angles A, B, C, &c. at the base, may be considered as parts of solid angles equal to P, formed by five plane angles, the part included by three of the plane angles being cut off by the plane angle of the base of the pyramid. On each side of the base of the pyramid let an equilateral triangle be constructed, so that its plane shall be inclined to the

adjacent lateral face of the pyramid at the same angle as any two of the adjacent lateral faces; that is, so that the angle under the planes A B C′ and A B P shall be equal to the angle under any two adjacent planes containing the angle P, and so that the same may be true of the planes B C D′ and B C P, C D E′ and C D P, &c.

Hence it follows, that at each of the vertices A, B, C, &c. of the base of the pyramid there are four angles, each two thirds of a right angle, and whose planes are united at the same inclinations as four of the angles which form the solid angle P. It follows, therefore, that the angle C′ B D′ included between the conterminous sides (B C′, B D′) of two equilateral triangles A B C′, C B D′, constructed upon conterminous sides of the pentagonal base, must be an angle of an equilateral triangle, so placed that if its plane be supposed to be drawn it will complete the solid angle B, and render it equal to P. The same conclusion is obviously applicable to each of the other angular points of the base.

We have thus a figure formed having a solid angle at P formed of five angles of equilateral triangles, having ten equilatural triangular faces, and a serrated edge or boundary A C′ B D′ C E′, &c., the planes of the angles being so disposed that if the gaps C′ B D′, D′ C E′, &c. be filled up, solid angles will be formed at A, B, C, &c. equal to P.

Let another figure in every respect equal and similar to this be formed, the corresponding points being marked by the small letters *a*, *b*, *c*, ... *a*′, *b*′, *c*′, &c. Let the point *c*′ be placed upon B, and the sides *c*′ *a*, *c*′ *b*, upon the equal sides B C′, B D′ of the equal angle C′ B D′. It is evident that the points *a* and *b* will coincide with C′ and D′ respectively. Thus the angle *a c*′ *b* inserted in C′ B D′ will complete the solid angle B, which will then be equal to P.

The plane of the angle D′ B C has been already proved to be inclined to that of D′ B C′ at the same angle as any two adjacent plane angles of P, and the

same is true of the planes of the angles $a\,c'\,b$ and $c'\,b\,d'$. Since, then, the plane $a\,c'\,b$ coincides with C′ B D′, and the planes $c'\,b\,d'$ and B D′ C are equally inclined to that plane, the plane $c'\,b\,d'$ must coincide with B D′ C. Since the line B D′ coincides with $c'\,b$, and the angles B D′ C and $c'\,b\,d'$ are equal, and in the same plane, the point d' must coincide with C. In the same manner we may prove that the points c, e', &c. coincide with E′ D, &c.; and we may prove that each of the solid angles at these points is equal to P, as we have already proved of the solid angle B.

Hence it appears, that by the union of the two shells formed of ten equilateral triangles, in the manner already described, a regular solid with twenty triangular faces is formed.

This solid is called the *regular icosaedron*.

(540.) By the construction it appears, that the inclinations of the planes of every pair of adjacent faces are equal. To determine this inclination conceive lines drawn from any two vertices A, C to the middle point of the opposite edge B P. These two lines being perpendicular to B P will contain an angle equal to the inclination of the planes A P B, C P B. But they are the sides of an isosceles triangle, whose base is the diagonal A C of the regular pentagon, and they are each equal to the altitude of an equilateral triangle, whose side is one of the edges. Hence the inclination of the planes of the faces of a regular icosaedron is equal to the vertical angle of an isosceles triangle, whose base is to its side as the diagonal of a regular pentagon to the altitude of an equilateral triangle constructed on one of its sides.

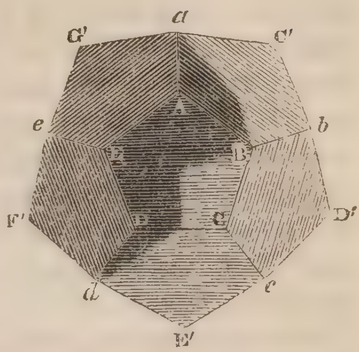
(541.) To construct a regular solid with square faces. This is obviously a rectangular parallelopiped, whose base is a square, and whose altitude is equal to the side of the base.

The *regular hexaedron* is therefore the cube.

(542.) To construct a regular solid with pentagonal

faces. Let A B C D E be a regular pentagon. From the vertex A draw the line A a equal to the side of the pentagon, and inclined to A B and A E at angles equal to the angle of the pentagon. The solid angle formed by the three lines which meet at that point is one of the angles of the required solid, formed by the three pentagonal angles a A B, a A E, and B A E. In the same manner, let the lines B b, C c, &c. be drawn from each of the angles of the pentagon, forming solid angles of the same kind at the points B, C, D, &c. Let the pentagon, of which a A B b are three sides, be completed, and in the same manner let each of the other pentagons on the sides of the base A B C D E be completed. We shall thus have a shell with six regular and equal pentagonal faces, and a serrated edge, a C′ b D′ c, &c. The adjacent planes, forming several pentagonal faces, are inclined each to each at the same angle; and it may be proved in the same manner as in (539.), that if a plane be drawn through the angle C′ b D′, a solid angle will be formed at b equal to those at A, B, C, &c. As in (539.), let another shell in every respect equal and similar to this be constructed, and let them be united at their serrated edges. It will follow, by the reasoning used in the former case, that the several solid angles which will be formed at a, C′, b, D′, &c. will be equal to those at A, B, C, &c.

fig. 199.



Hence, by the union of those two shells with six pentagonal faces, a regular solid with twelve pentagonal faces is formed.

This solid is called the *regular dodecaedron*.

(543.) To determine the inclination of the planes of the adjacent faces. Let any edge B A be conceived to be produced through A, and from a and E let perpendiculars to it be drawn in the planes of the angles B A a

and B A E. Since the angles B A a and B A E are equal, those perpendiculars will meet B A produced in the same point, and will include an angle equal to the inclination of the faces B A C′ and B A D. The diagonal a E will be the base of an isosceles triangle, of which the perpendiculars are sides. Hence the inclination of the faces is the vertical angle of an isosceles triangle, whose base is to its side as the diagonal of a regular pentagon is to the perpendicular from one of its angles upon a side terminated at the adjacent angle.

(544.) The volumes of all the regular solids are found by methods similar in principle to those which have been explained for solids in general. Each of these bodies admits of being circumscribed by a sphere, whose surface shall pass through the vertices of all its angles; if the centres of its faces be taken, and perpendiculars be raised from them, these perpendiculars will all pass through the centre of the circumscribed sphere. If the planes of its faces be produced, they will intersect the sphere, and their sections with it will form lesser circles of the sphere, and will be the circles circumscribing the regular polygons that form its faces: the centres of these latter circles will be the centres of the polygons; and it is plain, therefore, that the perpendiculars from them must all pass through the centre of their sphere. If lines be drawn from the centre of the sphere to the angles of the polyedron, these lines will be the edges of regular triangular pyramids, whose bases will be the faces of the figure, and the volume of the solid will be the sum of the volumes of such pyramids; or since they are all equal, it will be the volume of one of them multiplied by the number of faces which the solid has. Perpendiculars drawn from the centre of the sphere to the several faces of the solid will be equal, and a sphere described with the centre of the solid for its centre, and such a perpendicular for its radius, will touch all the faces of the solid at their respective centres, and will therefore be the sphere inscribed in the solid.

(545.) The volume of the solid will then be equal

to its entire surface multiplied by $\frac{1}{3}$ of the radius of the inscribed sphere. In the following table, the surfaces and volumes of the five regular solids whose edges are the linear unit are given.

No. of Sides.	Name.	Surface.	Volume.
4	tetraedron -	1·7320508	0·1178513
6	hexaedron -	6·0000000	1·0000000
8	octahedron -	3·4641016	0·4714045
12	dodecaedron	20·6457288	7·6631189
20	icosaedron -	8·6602540	2·1816950

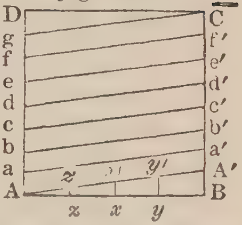
(546.) Since the volume is equal to the surface multiplied by $\frac{1}{3}$ of the radius of the inscribed sphere, that radius may be found by dividing 3 times the volume by the surface.

CHAPTER XIX.

ON HELICES AND SCREWS.

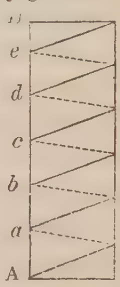
(547.) Let A B C D (*fig.* 200.) be a rectangular sheet of paper, and let A D be divided into a number of equal parts at a, b, c, d, e, f, g, and let B C be similarly divided at A′, a′, b′, c′, d′, e′, f′, and let the lines A A′, a a′, b b′, &c. be drawn.

fig. 200.



If the paper be now wrapped round a right cylinder, the circumference of whose base is equal to A B, the edge A D of the paper coinciding with the side of the cylinder, will exactly meet the edge B C. The point A′ will coincide with *a*, the point *a′* with *b*, the point *b′* with *c*, and so on. The line A A′ winding round the cylinder will meet the line *a a′*, at *a*, (*fig.* 201.) and both these lines being equally inclined to a section of the cylinder at *a* parallel to its base, they will form one continued line round the cylinder without making any angle.

fig. 201.

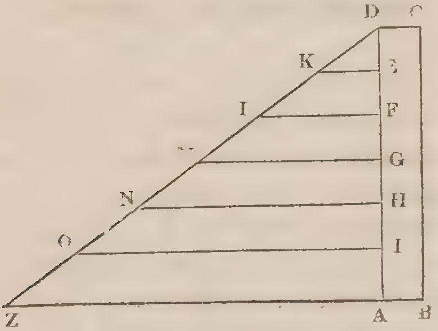


(548.) The line thus formed on the cylindrical surface, is a curve called a *Helix*. If the line A B (*fig.* 200.) be divided into any number of equal parts, at *z*, *x*, *y*, &c. the perpendiculars *z z′*, *x x′*, *y y′*, &c. will be proportional to their distances from A, because of the similarity of the right angled triangles of which these lines are the bases, and of which A is the common vertex. When these points *z′*, *x′*, *y′*, &c. are transferred to the cylindrical surface, their distances from the base of the cylinder will be proportional to that part of the circumference of the base which lies between the point A

and the perpendicular itself. The helix therefore may also be conceived to be traced on the cylindrical surface by a point which, while it moves uniformly round the cylinder, has also a motion parallel to its axis.

(549.) The helix may also be conceived to be produced in the following manner: let A B C D, (*fig.* 202.)

fig. 202.



be a cylinder, and let A Z be the base of a right-angled triangle, whose perpendicular A D is made to coincide with the side of the cylinder. Let the parallel E K be equal to the circumference of the cylinder, and supposing the points F, G, H, I, to divide the side of the cylinder into equal parts, the parallel F L will be twice the circumference of the cylinder, G M three times the circumference of the cylinder, and so on.

If the paper forming the right-angled triangle D A Z be now conceived to be rolled round the cylinder, a spiral curve will be formed upon its surface by the line D Z. After the paper has made one revolution of the cylinder, the point K will fall upon E. After the second revolution, the point L will fall upon F. After the third revolution, the point M will fall upon G, and so on.

(550.) The spiral line thus traced on the cylinder, is called the *thread* of the helix, and the distance D E or E F between the parallels is called the distance between two contiguous threads.

The angle Z is the angle under the thread of the helix and the base of the cylinder.

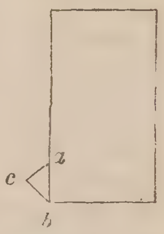
(551.) It is evident that for the same helix the distance between the successive revolutions, or between the

contiguous threads, is the same throughout the whole length of the cylinder.

(552.) The same helix may be formed on the cylinder in contrary directions, according to the direction in which the triangle D A Z is rolled on the cylinder.

(553.) If, instead of a point being moved along the helix on a cylindrical surface, any plane rectilinear figure be so moved, its plane being preserved so as constantly to pass through the axis of the cylinder, a spiral channel or tube will be formed, the section of which, by a plane through the axis of the cylinder, shall be equal to the rectilinear figure so moved. Thus, if a triangle, such as *a b c* (*fig.* 203.), be moved, so that its base *a b* shall always coincide with the surface of the cylinder, its plane passing through the axis of the cylinder, and the point *c* tracing on the cylinder a helix, the triangle will form on the cylinder as it moves the thread of a screw, and if such a thread be so formed in relief on the cylinder, the cylinder will become an ordinary screw with a triangular thread. Such a screw is called a *convex* or *male screw*.

fig. 203.



(554.) If the triangle be similarly moved on the concave surface of a hollow cylinder, its base, coinciding with the side of the cylinder, and its vertex always pointing through the axis, a similar screw will be formed, having a triangular thread sunk on its surface. Such a screw is called a *concave* or *female screw*.

(555.) A square might, in like manner, be moved along the direction of a helix, so as to form a screw with a thread of a corresponding form.

(556.) If a concave and convex cylinder have the same diameter, so that one may move within the other, and a similar and equal screw be formed on each, the raised thread of the convex screw will be capable of moving in the sunk thread of the concave screw, and if

either be made to revolve round the common axis of the cylinders, the other being at the same time prevented from revolving, the one cylinder will move within or round the other, and in each revolution will advance through a space equal to the distance between two contiguous threads of the screw.

(557.) If, in this case, the concave screw be kept in a fixed position, being prevented from moving either progressively or round its axis, the convex screw will, when it revolves, have a progressive motion, the speed of which will be to the speed with which its surface revolves, as the distance between the contiguous threads is to the circumference of the base of the cylinder.

(558.) If, on the other hand, the convex screw be kept fixed, being prevented from moving either progressively or by rotation, the revolution of the concave screw will impart to it a progressive motion, the speed of which will be determined in the same manner.

(559.) One of the screws may be capable of a progressive motion only, while the other is capable only of a motion of revolution; in that case the revolution of the one will impart a progressive motion to the other, and the rate of such progressive motion will be determined as above.

(560.) In virtue of this property, the screw is used in machinery as a means of converting a rotatory motion into a progressive motion; and it is especially applicable where the velocity of the progressive motion intended to be produced, is small compared with that of the rotatory motion which produces it.

(561.) In mechanics, the intensity or energy with which a force acts, increases as the space in which the action takes place is diminished. By this mechanical law, the screw becomes an agent of great power when a force of great intensity is required to be exerted through a small space. Screw-presses derive their efficacy from this principle. To the cylinder of the screw, and at right angles to it, is attached a handle, bearing com-

monly at its ends heavy spheres of metal, while, under the lower end of the screw, is placed the body on which a pressure is to be exerted. The cross handle and heavy spheres being made to revolve with considerable velocity, the screw descends with a progressive motion slower than that of the spheres, in the ratio of the circumference described by the spheres to the distance between the threads of the screw, and it acts upon the body under it with an energy greater in the same proportion.

(562.) Screws constructed with extremely fine threads are used as instruments for measuring extremely small magnitudes, and are thence called *Micrometer screws.* These screws are of considerable use in astronomical instruments, where spaces are required to be measured so minute that they cannot be seen without the aid of microscopes. These spaces are usually divided by a series of fine wires extended parallel to each other across the field of view of the microscope. One of these wires is capable of being moved parallel to itself, and made to approach to, or recede from, the other. If such a wire is made to coincide successively with two points, the distance between which it is required to measure, that measurement will be effected, if by any means the space through which the wire is moved can be known.

This is accomplished by putting the frame containing the movable wire in connection with a micrometer screw, so that the frame and wire shall be moved in the one direction or the other, by turning the screw. In this manner, each revolution of the screw moves the wire through a space equal to the distance between its threads, and any fractional part of a revolution will move the wire through the same fractional part of the distance between the threads. Thus, if the screw be cut with such a degree of fineness, that there shall be 100 threads in an inch, then each revolution of the screw will move the wire through the hundredth part of an inch, and the hundredth part of a revolution of the

screw will move the wire through the ten-thousandth part of an inch. The fractional parts of a revolution may easily be noted by placing an index or hand on the head of the screw, which shall play upon a graduated circle, divided according to the accuracy of the intended observation.

(563.) The application of screws in the arts as adjusting screws is frequent; in this case less accuracy of construction is required. If an instrument, for example, supported on three or more legs, is required to be levelled, a screw is fixed in each leg, by turning which the level of the instrument is gradually adjusted.

(564.) In the art of distillation, the vapour raised from the liquid to be distilled is conducted through a *worm*, which is nothing more than a tube bent into the form of a helix, and immersed in a cistern of cold water. The steam, or vapour, passing through this worm, is deprived of its heat, and reconverted into liquid, or condensed, and drops from the lower end into a vessel intended to receive it.

(565.) The screw by which corks are drawn from bottles, is a steel wire bent into the form of a helix, and sharpened at the point. This instrument penetrates the cork, and forms through it a hollow path, likewise in the form of a helix, and as it revolves advances downwards, moving through a depth equal to the distance between the threads or spires in each revolution of the screw.

(566.) The plaits of straw by which hats are formed are carried round the circumference of the hat in the form of a helix; the distance between the threads being equal to the breadth of the plait. In proportion as the plait is of uniform breadth, and accurately united, edge to edge, so will the fabric be the more perfect. This constitutes the superiority of the Italian bonnets.

(567.) Steel wire bent into the form of a helix, and rendered highly elastic, is much used in the arts for springs. The common spring steel-yard is an instru-

ment formed of an elastic spring in the form of a helix confined within a cylinder. The matter to be weighed is suspended from a hook, so that its weight shall compress the spring, and the extent of such compression shows the amount of the weight.

(568.) The coaches which form a railway train are liable, when the train is suddenly stopped or retarded, to strike one against the other, with such a force as to be attended with injurious consequences to the passengers, and in the event of one train overtaking another the collision is still more dangerous. These effects are mitigated by attaching to the ends of the carriages circular cushions called *buffers,* which are fastened to iron rods that pass under the carriage, and act against a system of elastic springs. When one carriage encounters another, these buffers come first in contact one with another, and the force of the collision is broken by the elasticity of the springs. The springs used in some carriages for this purpose have the form of a helix, that being the spring which has most longitudinal play.

(569.) The form of the helix is sometimes presented in natural objects. The tendrils of creepers and parasite plants frequently take this form, winding round the trunk of the larger tree which forms their support. The tresses of the human hair are sometimes elastic spirals or helices, and this form, being admired, is accordingly imparted to them by artificial means. The fibres constituting threads or ropes are, by the process of spinning or twisting, thrown into the form of the helix.

(570.) If a vertical line be conceived to be the axis of a cylinder, and from any point in it a horizontal line equal to the radius of the cylinder be drawn, and this horizontal line be supposed to ascend with a uniform motion along the vertical line, and at the same time to revolve with a uniform motion round it, the end of the horizontal line will trace a helix on the cylindrical surface, and the line itself, as it ascends and revolves, will trace a helical surface round the axis of the cylinder.

This spiral surface might also be conceived to be formed by drawing radii of the cylinder from every point of a helix described upon its surface to its axis.

(571.) Such a spiral surface is the form of spiral staircases, sometimes called geometrical staircases. They are usually constructed within pillars or towers, and are the means of ascending them.

(572.) A hollow cylinder, with such a spiral surface constructed within it, is called the *screw of Archimedes*, that mathematician having been its inventor.

CHAP. XX.

OF THE INTERSECTIONS OF SURFACES.—OF THE CONIC SECTIONS.

(573.) As all surfaces may be generated by the motion of lines restricted by an infinite variety of conditions, so all lines may be produced by the intersection of surfaces under circumstances equally various. In fact, all the lines which are produced, or really exist, in natural or artificial objects, are formed by the intersection of surfaces forming corners or edges.

(574.) If two plane surfaces intersect, their line of intersection will, as has been already explained, be a straight line. Consequently, all the edges of solid figures, whose faces are plane, must be straight lines.

(575.) Although the intersection of a plane surface with a curved surface, or of two curved surfaces with each other, is not in general a straight line, it must not therefore be inferred that it is never so. On the contrary, the intersections of a cylindrical surface, with a plane parallel to its axis, are parallel right lines; and the intersection of a conical surface, with a plane passing through the vertex of the cone, are right lines intersecting at the vertex. In fact, any surface which can be generated by the motion of a right line—or, in other words, any developable surface—will be intersected in a right line by a plane which passes through it in the direction of the line by the motion of which it is generated.

(576.) In like manner, two cylindrical surfaces whose axes are parallel, or two conical surfaces which have a common vertex, will intersect each other in straight lines, the intersections of the former being straight lines pa-

rallel to the axes of the cylinders, and the intersections of the latter being straight lines passing through the common vertex of the two cones.

(577.) And, in general, two developable surfaces will intersect in a right line, if the right lines, by the motion of which they are generated, coincide in any one position.

(578.) But these are the exceptions, being the peculiar and the only conditions under which curved surfaces intersecting each other, or intersecting plane surfaces, can produce a right line. In general, the line produced by their intersection will be a curve, the nature and properties of which will depend on the form and position of the intersecting surfaces.

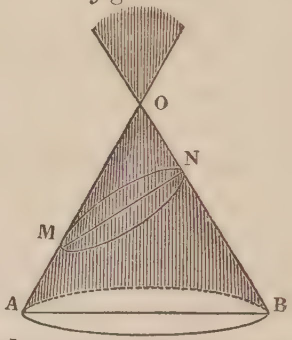
(579.) It has been already shown, that all surfaces of revolution, intersected by a plane perpendicular to their axis, have circular sections ; and it likewise follows, that any two surfaces of revolution, intersecting each other, will have a circle for their common intersection if they have a common axis.

(580.) The curves formed by the intersection of a plane with a curved surface, which have been of the greatest importance by reason of their use in the arts and sciences, and of the greatest intellectual interest by reason of the beauty of their forms and properties, are those which are formed by the intersection of a plane with the surface of a cone. When in the infancy of science the investigation of the properties of these curves was pursued by Plato, and the geometers of his school, as matter of pure and sublime intellectual speculation, the reproach of inutility was cast on such inquiries, as it is now frequently, and with as much ignorant presumption, advanced against the investigations of the higher analysis. The possibility was not then foreseen, that the progress of discovery, after two thousand years had rolled away, would ultimately establish the fact, that these very curves, which were regarded by the disciples of Plato in nearly the same light as abtruse metaphysical speculations are now viewed, are the paths in which the earth and planets move round

the sun; in which the satellites move round their primaries; and are even the forms to which these great bodies of the universe themselves are reduced by the forces which attend their rotation on their axis.

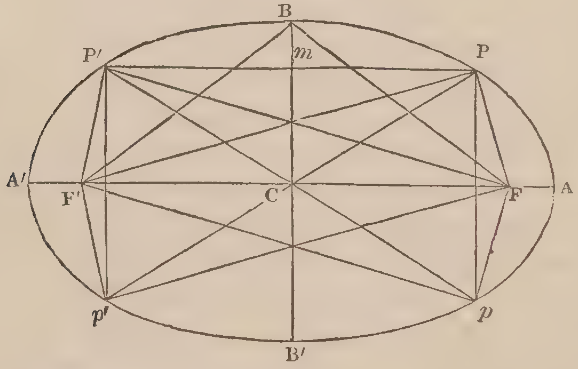
(581.) If a cone A O B (*fig.* 204.) be cut obliquely by a plane which intersects two sides of the angle, by the revolution of which the cone is produced, at two points M and N, which are at the same side of the vertex O of that angle, the section will be the curve called an *ellipse*.

fig. 204.



This curve may be described upon a plane in the following manner. Let two pins be attached to two points F and F′ (*fig.* 205.), and to these pins let the

fig. 205.



ends of a thread F P F′ be fastened, the length of the thread being greater than the distance between the pins. Let a pencil be looped in the thread, by which it shall be extended so as to form two sides of a triangle, of which the distance between the pins shall be the base. Thus placed, let the pencil be moved in the loop of the thread, keeping the thread constantly stretched. The sides of the triangle formed by the thread will vary their lengths, one increasing by as much as the other diminishes. As the pencil is moved downwards it will trace the curve

P A, and when it attains the point A, the thread will be doubled upon the line F A, the single thread only extending over F′ F, so that the sum of the lines F′ A and F A being equal to the length of the thread, will be equal to the sum of the sides F′ P and F P of the triangle, in every position which the pencil can assume. As the pencil is moved to the left, it will trace the curve P B, and will attain its highest position B, when the sides F B and F′ B of the triangle formed by the thread shall become equal; hence if F F′ be bisected in C, and B C joined, B C will be at right angles to F F′. If the pencil be moved to the left of B and carried downwards, the sides of the triangle will undergo precisely the same changes of magnitude as they would in moving the pencil from B to A, only that the lesser side of the triangle will be terminated at F′, instead of the greater side. The pencil in two corresponding positions is represented at P and P′, the two triangles F P F′ and F P′ F′ being in all respects equal, but reversed in position. If the quadrant of the curve B A were doubled over on B A′, forming a fold along B C, the line C A would fall on C A′, and the point P would fall on the point P′; and, in the same manner, it may be shown that every part of the curve, from B to A, would coincide with the curve from B to A′. The quadrant, therefore, of the ellipse from B to A is perfectly equal and similar to the quadrant from B to A′, and the line B C divides the semi-ellipse symmetrically. All lines such as P P′ parallel to A A′, and therefore perpendicular to B C, will be bisected by B C.

If the thread be now stretched below the line F F′, and the pencil be moved in it in a similar manner, a curve will be formed below the line F F′, in all respects equal and similar to the curve A B A′ above it; the generating triangle, as the pencil moves, will undergo the same changes, the pencil taking successively positions below the line F F′, similar to those which it previously took above that line: thus the points *p*, *p*′, below the line F F′, will correspond in their position to the points P, P′ above that line, and the lines P *p* and P′ *p*′ will be

perpendicular to A A′, and therefore parallel to B B′, and will be bisected by A A′.

(582.) The lines A A′ and B B′ are called *axes* of the ellipse. The line A A′ is called the *transverse axis*, and the line B B′ the *conjugate axis.*

(583.) From what has been proved, it is evident that each of the axes divides the ellipse symmetrically, and that a system of chords, perpendicular to either axis, are bisected by that axis.

(584.) When a system of parallel chords are all bisected by the same straight line in any curve, that line is called a *diameter* of the curve, and the halves of the chords so bisected are called the *ordinates* to that diameter.

(585.) When the ordinates of a diameter are at right angles to it, the diameter is called an *axis* of the curve.

(586.) Since every diameter of a circle bisects a system of chords perpendicular to it, all diameters of a circle are axes.

(587.) The point C, where the axes of the ellipse intersect, is called the *centre* of the ellipse. If the line P C be produced to meet the ellipse in the opposite quadrant at p', the point p' will have the same position in the quadrant A′ B′ as the point P has in the quadrant A B, and the triangle F p' F will be in all respects equal to the triangle F′ P F. It will be evident therefore, that the line P p' will be bisected at C; and in the same manner, all lines drawn through the point C, and terminating in the ellipse, may be shown to be bisected at C. It is from this property that the point C has been called the centre of the ellipse.

(588.) The points where an axis of a curve meets it, are called *vertices* of the curve.

(589.) The ellipse has therefore four vertices, A, A′, B and B′.

(590.) As the points in the circumference of an ellipse possess the character of being so placed, that the sum of their distances fròm the two points F and F′

has everywhere a given length, any point, the sum of whose distances from F and F′ is greater than this given length, will be outside the ellipse; and any point, the sum of whose distances from F and F′ are less, will be within the ellipse.

(591.) When the pencil is moved to the point A, the thread will be extended from F′ to A, and from A to F; therefore A F′, together with A F, will be equal to the length of the thread; but A F is equal to A′ F′, therefore the length of the thread will be equal to A A′, the transverse axis of the ellipse.

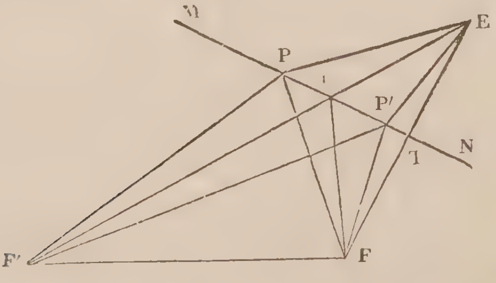
(592.) Hence the sum of the distances of any point in the ellipse, from the points F and F′, is equal to the transverse axis.

(593.) The points F and F′ are called the foci of the ellipse.

(594.) Since the vertex B of the conjugate axis is equally distant from the foci, its distances from the foci will be equal to half the transverse axis.

(595.) To explain the method of drawing a tangent to a given point in the ellipse, we shall first show that if from two given points F and F′ (*fig.* 206.) lines be drawn to the same point P in another line M N, so as

fig. 206.



to make equal angles with it, their sum will be less than the sum of the lines drawn to any other point, such as P′, in the same line M N.

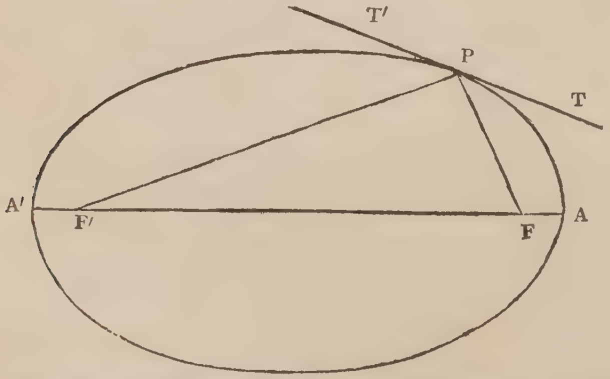
From F draw F T perpendicular to M N, and produce it, so that T E shall be equal to T F, and draw F′ E. From the point P, where this line meets M N,

draw P F, the lines F F and P F′ will then be equally inclined to the line M N. For, since the line P T is perpendicular to F E and bisects it, the triangle F T P is equal in every respect to the triangle E T P, therefore the angles F P T and E P T are equal; but the angle E P T is equal to the angle F′ P M, therefore the angle F P T is equal to the angle F′ P M, that is to say, the lines F P and F′ P are equally inclined to the line M N.

Let P′ be any other point on the line M N, and draw P′ F, P′ F′, and P′ E. From the identity of the triangles F P T and E P T, we have P F equal to P E, and for a similar reason P′ F is equal to P′ E; the line F′ E will therefore be equal to the sum of the distances F′ P and F P, and the lines F′ P′ and P′ E will be equal to the sum of the distances F′ P′ and P′ F; but since F′ P′ and P′ E are together greater than F′ E, the sum of the distances of P′ from F′ and F, which is equal to the former, will be greater than the sum of the distances of P from F′ and F, which is equal to the latter. The sum of the distances, therefore, of P from F′ and F, is less than the sum of the distances of any other point in the line M N from the points F′ and F.

(596.) To draw a tangent at a point P in an ellipse. From the foci F and F′ (*fig.* 207.), draw lines to the

fig 207.



point P, produce F′ P to E (*fig.* 206.) making P E equal to P F, then P T drawn perpendicular to E F will make equal angles with the lines P F and P F′. This line will

be a tangent to the ellipse at P; for, by what has been already proved, the sum of the distances of the point P from the foci is less than the sum of the distances of any other point in the line T T′ from the foci. Therefore, every point in that line, except the point P, must lie outside the ellipse, and, therefore, the line T T′ is a tangent to the curve.

(597.) As the curve coincides in direction with its tangent, it appears that right lines from the foci to any point in the ellipse are equally inclined to the ellipse; and if a spheroid be generated by the revolution of the ellipse round its transverse axis A A′, all lines from the foci to any point in the surface of this spheroid will be equally inclined to that surface.

(598.) Hence arise some remarkable and beautiful physical properties of spheroidal surfaces of this kind.

(599.) It is a well-known property of rays of light, that when they strike upon any reflecting surface, they will be reflected from that surface, in directions inclined to it, at the same angle as that at which the incident ray is inclined to it. Thus, if F (*fig.* 207.) were a luminous point, and F P a ray of light proceeding from it, that ray of light would be reflected from it in the direction P F′. If, therefore, a luminous object be placed in one focus of an elliptical spheroid, the rays diverging from it, after being reflected by the surface of the spheroid, will converge to the other focus; any object, therefore, placed in the other focus, would thus receive by reflection all the light proceeding from the luminous point.

(600.) Rays of heat being subject to the same law would be similarly reflected; and, therefore, a heated body placed in one focus, of such an elliptical spheroid, will have its heat collected by reflection in the other focus. A red-hot ball, thus placed in the focus of an elliptical mirror, will set fire to an object placed in the other focus of the same mirror.

(601.) For the production of these effects, it is not necessary that the reflecting surface should form a com-

plete spheroid. If one or more reflecting surfaces be so placed as to form portions of the same elliptic spheroid, like effects would be produced; the quantity of rays, collected by reflection at the other focus, being proportioned to the extent of reflecting surface, which occupies the position of the surface of a spheroid.

(602.) Sound, propagated by the air, is reflected from smooth and even surfaces, according to the law which governs the reflection of light and heat. If a sound be produced in one focus of an elliptical spheroid, it will be heard at the other focus, at the end of the time which it takes to move through F F′, the distance between the foci; but, as it also will proceed from the sounding body in every direction around F, it will encounter the surface of the spheroid, and be reflected from it to the other focus F′. As the distance which each pulsation of sound will have to move through by reflection will be the same, being equal to the sum of the distances of the points in the ellipse from the foci, and as all the pulsations move with the same speed, all the reflected sounds will arrive at the same moment at F′, and if the reflecting surfaces are sufficiently extensive they will produce an effect sufficiently strong to be audible. A listener at F′ will, therefore, hear any sound produced at F twice; first, after the time which such sound would take to move from F to F′, and again, after the time it would take to move from F to P, and from P to F′. This repetition constitutes what has been called *echo*.

It is possible to conceive the echo of a sound produced in this way to be louder than the sound itself, or, to speak more correctly, that the sound heard by reflection shall be louder than the sound heard directly. A sound diminishes in loudness by increase of distance; the reflected sound would, on that account alone, be less loud than the direct sound, because the sum of the distances of a point in the spheroid from the foci, or, what is the same, the transverse axis of the spheroid is greater than the distance between the foci. But this cause of

diminished intensity may be more than compensated by the extent of surface from which the echo is reflected.

(603.) If two or more spheroidal surfaces, or parts of spheroidal surfaces, have the same foci, then any sound produced in one will be repeated as many times at the other as there are such surfaces, and the interval between the echos will be measured by the time that sound takes to move through a space equal to the difference between the transverse axes of those surfaces. Hence, the reason is apparent why echoes are so frequently heard among mountains and never on plains; and also why, among mountains, the speaker and the hearer must assume particular positions in order that the echo may be perceived. The faces of the precipices form the reflecting surfaces which are casually placed either exactly or nearly in an elliptical position, and that the desired effect may be produced, the speaker and hearer must occupy positions in the foci of the ellipse.

It is said that cells have been so constructed in prisons, that every sound uttered by the prisoner, even in a low tone, is reflected by surfaces placed for the purpose, into another apartment invisible to the prisoner, where it is heard by the jailor or other persons placed there for the purpose.

(604.) The less the distance between the foci F F′ is in proportion to the transverse axis A A′, the nearer the ellipse will approach, in its form, to a circle; the ratio of this distance to the transverse axis, or, what is the same, the ratio of F C to F B (*fig.* 205.), has thence been called the eccentricity of the ellipse.

(605.) Similar ellipses are those which have equal eccentricities.

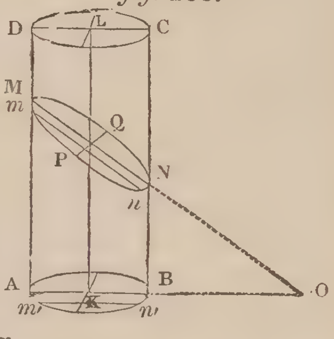
If in two ellipses the distances F C are proportional to F B (*fig.* 205.), the distance F B will also be proportional to B C, but the former being equal to half the transverse axis, it follows, that in similar ellipses the axes are in the same ratio.

(606.) If the foci F and F′ coalesce with the centre C by the distance between them vanishing, the ellipse

will become a circle. This change may be traced to the varying conditions arising out of the method of describing an ellipse, already explained. While the thread remains the same, the nearer the pins are brought to each other the more nearly will the ellipse approach to a circle in form; and when the pins are actually brought together, the pencil will describe a circle, of which half the length of the thread is the radius.

(607.) The ellipse has been stated to be formed by the section of a plane with a conical surface, but it may also be produced by the section of a cylinder with a plane. Let a right cylinder A B C D (*fig.* 208.) be intersected by a plane M N O oblique to its axis K L, and the section will be an ellipse; the transverse axis M N of which will be produced by its intersection with a plane through the axis K L of the cylinder, perpendicular to the line of intersection of the plane of the ellipse itself with the plane of the base of the cylinder.

fig. 208.



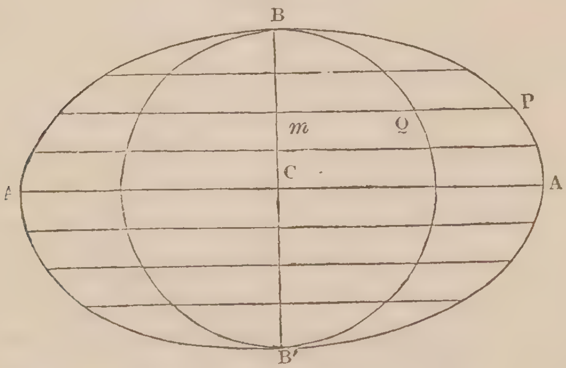
(608.) It is evident that the circular base of the cylinder is the orthographical projection of the ellipse on the plane of the base; the transverse axis M N of the ellipse is projected into the diameter A B of the circle, being diminished by such projection in the ratio of N O to B O; but the conjugate axis P Q of the ellipse, being parallel to the base of the cylinder, will not be diminished by projection, and will, therefore, be equal to the diameter of the base of the cylinder. All lines in the ellipse at right angles to P Q, such as $m\,n$, will be projected into corresponding lines, such as $m'\,n'$, at right angles to that diameter of the base which is parallel to P Q. All such lines will be reduced in the same proportion, that is, in the ratio of N O to O B.

(609.) Since the square of the ordinate to the diameter of a circle, is equal to the rectangle under the

segments of that diameter, and since the segments of P Q, the conjugate axis of the ellipse, are equal to the segments of the diameter of the base on which P Q is projected, it follows that the square of the ordinate to P Q will have to the rectangle under the segments into which the ordinate divides P Q, the same ratio as the square of that ordinate has to the square of its projection, or as the square of M N has to the square of A B, or, what is the same, as the square of M N has to the square of P Q. Hence, in general, the square of an ordinate, such as P *m* (*fig.* 205.) to the conjugate axis of an ellipse, is to the rectangle under the segments of the axis made by the ordinate, that is, the rectangle under B *m* and B′ *m*, as the square of the semitransverse axis A C, is to the square of the semi-conjugate axis B C.

(610.) If a circle be described on the conjugate axis of an ellipse as a diameter, as in *fig.* 209., the ordinates

fig. 209.



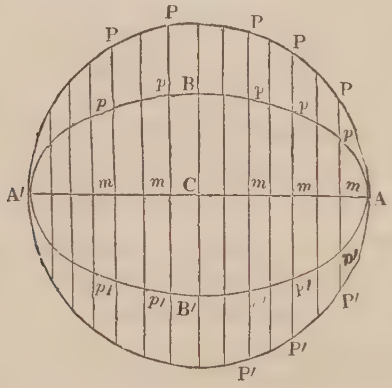
of the conjugate axis of the ellipse will all be divided in the same ratio by the circumference of the circle. For those parts of the ordinates of the ellipse intercepted within the circumference of the circle, correspond to the projections of the ordinates to the conjugate diameter P Q (*fig.* 208.) upon the base of the cylinder; the ratio of the ordinates Q *m* of the circle to the ordinates P *m* of the ellipse will therefore be that of the conjugate axis B B′ to the transverse axis A A′ (*fig.* 209.)

(611.) If a circle, whose plane is oblique to a horizontal plane, be projected by perpendiculars upon that plane, the projection will be an ellipse, whose transverse axis will be equal to the diameter of the circle, and whose conjugate axis will be less than that diameter, in the ratio of the side of a right-angle triangle to the hypotheneuse, that angle of the triangle adjacent to the side being equal to the angle of projection.

The projection of the centre of the circle will be the centre of the ellipse, and the projections of all ordinates to the horizontal diameter of the circle will be ordinates to the transverse axis of the ellipse; these ordinates of the circle will have to their projections, that is, to the ordinates of the ellipse, the same ratio as the diameter of the circle has to the conjugate axis of the ellipse.

(612.) Hence it follows, that if a circle be described on the transverse axis of the ellipse as a diameter, as in *fig.* 210., the ordinates to the diameter of the circle will

fig. 210.



be divided proportionally by the ellipse; therefore the ratio of P m to p m will be the same as the ratio of A C to B C.

(613.) If the area of the ellipse and circle be supposed to be divided into bands perpendicular to the transverse axis A A′, by ordinates P p m, placed so closely together that the arcs of the curves between them may be considered to be straight lines, the

areas of the spaces of the ellipse and circle between every pair of contiguous ordinates will be proportional to those ordinates, and as all the ordinates are in the same ratio, the sum of all the areas between the elliptical ordinates, that is, the area of the ellipse itself, will be to the sum of all the areas included between the circular ordinates, that is, to the area of the circle itself, as any one elliptical ordinate is to the corresponding circular ordinate, that is, as the conjugate axis of the ellipse is to the transverse axis. Hence the area of an ellipse is to the area of a circle, having its tranverse axis as diameter, as the conjugate axis of the ellipse is to the transverse axis.

(614.) Since it has been already proved, that a circle described on the conjugate axis, as diameter (*fig.* 209.), divides the ordinates to that axis proportionally, it may be shown, by reasoning similar to the above, that the area of the ellipse is to the area of the circle, having its conjugate axis as diameter, as the transverse axis is to the conjugate axis.

(615.) Hence, the area of the ellipse is a mean proportional between the circles described on its two axes as diameters.

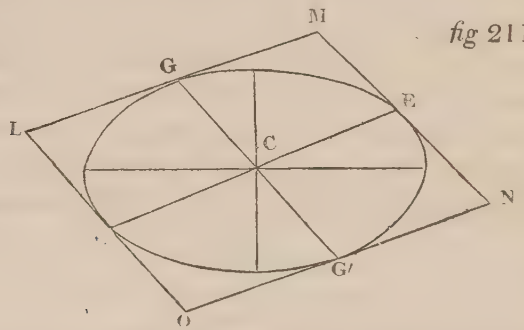
(616.) The area of an ellipse therefore is equal to the area of a circle, whose diameter is a mean proportional between its axes.

(617.) The proportion above established between the area of an ellipse and the areas of circles, having diameters equal to its axes, may also be shown by projection. The ellipse being the orthographical projection of a circle whose diameter is equal to its transverse axis, and whose plane is inclined at an angle to the plane of projection, determined in the manner explained in (611.), and the circle on its conjugate axis as diameter being the projection of an ellipse whose conjugate axis is equal to that diameter, and the position of whose plane is determined, as explained in (608.), the area of the first circle will be to that of the ellipse which is its projection, as the area of the ellipse is to the area of the second circle, which

is its projection, the angles of projection being the same in both cases.

(618.) If a square be circumscribed round a circle, and the circle be projected into an ellipse, the square will be projected into a parallelogram, and the diameters of the circle, joining the points of contact of the sides of the square, will be projected into diameters of the ellipse, joining the points of contact of the sides of the parallelogram, as represented in *fig.* 211. As the dia-

fig 211.



meters of the circle joining the points of contact of opposite sides of the square are parallel to the remaining sides of the square, their projections forming the diameters of the ellipse will be parallel to the sides of the parallelogram. Thus E E′ is parallel to the sides L M and N O, and G G′ is parallel to M N and L O.

(619.) Two diameters of the ellipse, such as E E′ and G G′, each of which is parallel to tangents through the extremities of the other, are called *conjugate diameters.*

(620.) Since all parallelograms formed by tangents, through the extremities of a pair of conjugate diameters, are the projections of a square circumscribing the circle of which the ellipse is the projection, it follows that the areas of all such parallelograms will have the same ratio to the area of that square, and their areas will therefore be equal.

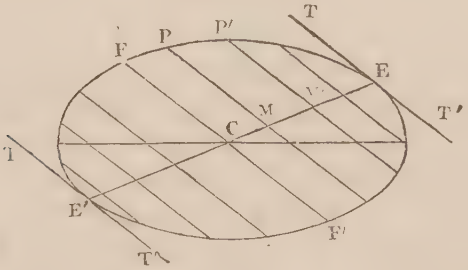
(621.) Since the area of the square, circumscribing the circle, bears the same ratio to its projection, that the area of the circle bears to that of the ellipse which is its projection; and, since the area of the circumscribing

square is the square of the transverse axis of the ellipse; it follows that the area of the parallelogram, which circumscribes any system of conjugate diameters of an ellipse, is to the square of the transverse axis of the ellipse, as the conjugate axis, is to the transverse axis.

(622.) The axes of the ellipse being themselves a pair of conjugate diameters, it follows that the area of a parallelogram, circumscribing any system of conjugate diameters, is equal to the rectangle under the axes.

(623.) Since every diameter of a circle bisects a system of parallel chords at right angles to it, and parallel to the tangents through its extremities, the projection of such diameter will be a diameter of the ellipse formed by the projection of the circle, and the ordinates to the diameter of the circle will be projected into a system of parallel chords of the ellipse (*fig.* 212.), bisected by the

fig 212.



diameter E E′ of the ellipse, which is the projection of the diameter of the circle; and the tangents at the extremities of the diameter of the circle will be projected into tangents T T′, at the extremities of the diameter of the ellipse.

(624.) Hence it appears, that every diameter of an ellipse bisects a system of chords, parallel to the tangents through the extremities of that diameter, which tangents are always parallel to each other.

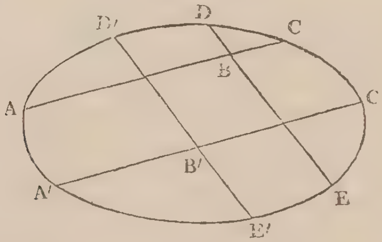
(625.) Since the squares of the ordinates to the diameter of the circle are equal to the rectangles under the corresponding segments of the diameter, the squares of the ordinates P M, to any diameter of an ellipse will be

proportional to the rectangle under the segments of such diameter; that is to say, the square of P M will be to the rectangle under E M and E′ M, as the square of P′ M′ is to the rectangle under E M′ and E′ M′.

(626.) Since the rectangle under the segments of E E′ corresponding to F C is the square of E C, it follows that the square of any ordinate P M to a diameter is to the rectangle under the segments of that diameter E M and E′ M, as the square of the semi-conjugate diameter F C is to the square of E C the semi-diameter itself.

(627.) Since the rectangles under any two chords of a circle intersecting each other are equal, and since parallel chords in the circle are proportional to their parallel projections in the ellipse, it follows that if two intersecting chords of an ellipse, such as A C, D E (*fig.* 213.), be parallel to two other intersecting chords,

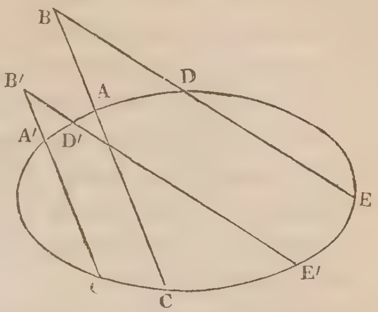
fig. 213.



such as A′ C′, D′ E′, then the rectangle under the segments of A C made by the point B is to the rectangle under the segments of D E made by the same point, as the rectangle under the segments of A′ C′, made by the point B′, is to the rectangle under the segments of D′ E′ made by the same point.

(628.) Since in a circle, the rectangles under the parts of secants drawn from the same point outside it, between that point and the circumference, are equal, the rectangles under the corresponding parts of parallel secants to an ellipse, which are the projections of the former, will be proportional: thus, if B E and B C (*fig.* 214.) be

fig. 214.



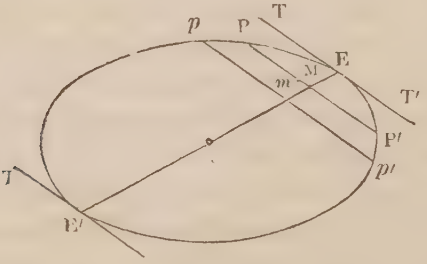
parallel to B′ E′ and B′ C′, then the rectangle under B D and B E will be to the rectangle under B A and B C as the rectangle under B′ D′ and B′ E′ is to the rectangle under B′ A′ and B′ C′.

(629.) Since diameters of the ellipse bisect each other, the rectangles under the segments, made by their mutual intersection, will be the squares of their halves; hence it follows, by what has been just proved, that the rectangles under the parts of intersecting chords or secants, between their common intersection and the circumference of the ellipse, are proportional to the squares of the semi-diameters to which they are parallel.

(630.) As every system of parallel chords in a circle have their middle points placed on a diameter of that circle to which they are ordinates, so every system of parallel chords in an ellipse, being the projection of the former, will have their middle points situate on a diameter of the ellipse which is the projection of the latter.

(631.) Hence a tangent may be drawn to an ellipse which shall be parallel to any given line; for, let two chords P P′ and $p\,p'$ (*fig.* 215.), be drawn in the curve

fig. 215.



parallel to the given line, and let them be bisected at M and m, and through these points of bisection let a line

E E′ be drawn; a line T T′ drawn through E parallel to P P′, will be the tangent required: for the chords P P′ and $p\,p'$, being bisected by E E′, will be ordinates, and E E′ will be their corresponding diameter. The lines T T′, therefore, drawn through the extremities of this diameter, are the required tangents.

(632.) To find the centre of a given ellipse.

Draw any two parallel chords, such as P P′ and $p\,p'$ (*fig.* 215.), and bisect them; the line E E′ passing through their points of bisection, will be a diameter, and its point of bisection C, will be the centre.

(633.) Given a diameter E E′ (*fig.* 216.) in an ellipse; to find its conjugate diameter.

Draw any chord of the ellipse $e\,e'$ parallel to E E′, and bisect it. Through its point of bisection m, draw the diameter F F′; this diameter will be conjugate to the diameter E E′, since its ordinate $e\,e'$ is parallel to E E′.

(634.) All diameters of an ellipse, which are inclined at equal angles to its axis, are equal.

For if C E and C E′ (*fig.* 217.) be two such semi-

fig. 216.

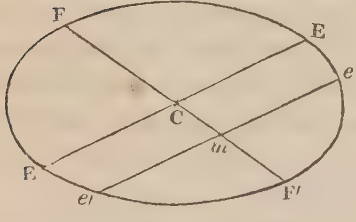
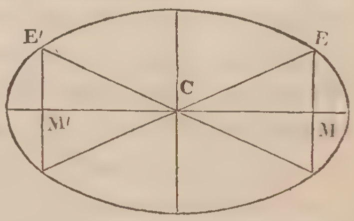


fig. 217.

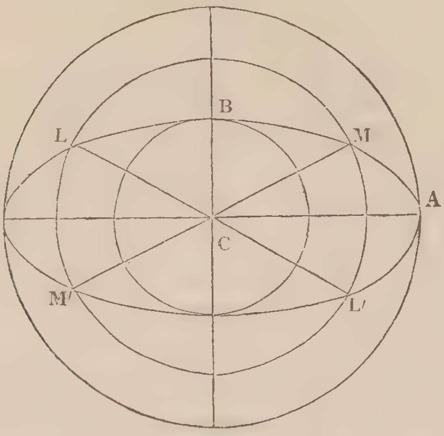


diameters, they will be terminated at points holding corresponding positions in the elliptical quadrants, so that the ordinates to the transverse axis passing through these points shall be equal. Since E M is therefore equal to E′ M′, and the angles at C are equal, E C will be equal to E′ C.

(635.) If a circle be described with the centre C as centre (*fig.* 218.) and any line greater than C B and less than C A as radius, such circle will be included between the circles described on the two axes as diameters, and

will consequently intersect the ellipse in four points, L, M, L′, M′; and, as these points will be equally distant,

fig. 218.



from the centre C of the ellipse, the diameters L L′ and M′ M will be equal and equally inclined to the axes of the ellipse.

(636.) Hence when an ellipse is given, its axes may be found.

For let its centre be found by (632.), and with its centre as centre, and any line drawn from it to the ellipse as radius, let a circle be described. If such circle touch the ellipse and lie entirely within it, its diameter through the points of contact will be the conjugate axis of the ellipse, and the diameter, at right angles to that, will be the transverse axis.

If it touch the ellipse, containing the ellipse entirely within it, its diameter through the points of contact will be the transverse axis, and a diameter at right angles to it will be the conjugate axis.

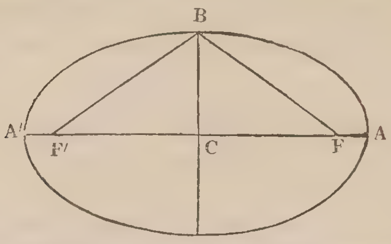
If it do not touch the ellipse it will intersect it in four points, and diameters through these points being drawn, those diameters of the ellipse which bisect the angles under such diameters will be the axes of the ellipse.

(637.) When an ellipse is given, to find its foci.

Find the axis by the method already explained, and from the vertex B of the conjugate axis, inflect upon

the transverse axes two lines B F and B F′ (*fig.* 219.),

fig. 219.



equal to the semi-transverse axis; the points F and F′ will then be the foci.

(638.) The distance C F of the foci from the centre is a line whose square is equal to the difference between the squares of the semi-axis.

(639.) Since the square of the semi-conjugate axis B C is equal to the difference between the squares of B F and F C, or of A C and F C, it is equal to the rectangle under A F and A′ F, that is, to the rectangle under the segments of the transverse axis made by the focus.

(640.) Among the properties of the ellipse which have been demonstrated in the preceding pages, there are some which admit of being expressed with greater clearness and conciseness by the symbols and notation of algebra.

Let A = the semi-transverse axis.

B = the semi-conjugate axis.

c = the distance of either focus from the centre.

$$e = \frac{c}{\mathrm{A}}$$

y = an *ordinate* P m (*fig.* 210.) to the transverse axis.

x = the corresponding *abscissa* C m, or part of the axis between this ordinate y and the centre C.

By (626.) it appears that

$$\frac{y^2}{(\mathrm{A}+x)(\mathrm{A}-x)} = \frac{\mathrm{B}^2}{\mathrm{A}^2},$$

$$\therefore \frac{y^2}{\mathrm{A}^2 - x^2} = \frac{\mathrm{B}^2}{\mathrm{A}^2},$$

$$\therefore \mathrm{A}^2 y^2 + \mathrm{B}^2 x^2 = \mathrm{A}^2 \mathrm{B}^2.$$

This equation, which expresses algebraically the relation

between the ordinates and abscissæ, referred to the axes of the ellipse, is called the *equation of the ellipse*, related to its axes. From this equation, by the ordinary operations of algebra, all the properties of the ellipse may be deduced.

If y express an ordinate P M to any diameter E E′ (*fig.* 215.), and x the corresponding abscissa C M, and the semi-diameter C E be expressed by A′, and its semi-conjugate diameter by B′, we shall have by (626.)

$$\frac{y^2}{(A' + x)\ (A' - x)} = \frac{B'^2}{A'^2},$$

$$\therefore \frac{y^2}{A'^2 - x^2} = \frac{B'^2}{A'^2},$$

$$\therefore A'^2 y^2 + B'^2 x^2 = A'^2 B'^2.$$

The equation of the ellipse, related to any system of conjugate diameters, has therefore the same form as when related to its axes.

Let φ = the angle E C G (*fig.* 211.) contained by any system of conjugate diameters.

The sides L M and L O of the parallelogram L N, formed by tangents through their extremities, will be equal to 2 A′ and 2 B′, and the area of such parallelogram will be $4\ A'\ B' \sin \varphi$. From what has been proved in (622), it follows that

$$4\ A'\ B' \sin \varphi = 4\ A\ B,$$

$$\therefore A'\ B' \sin \varphi = A\ B.$$

By (639.) it appears that

$$A^2 - B^2 = c^2 = e^2\ A^2,$$

$$\therefore B^2 = (1 - e^2)\ A^2.$$

Hence the equation of the curve related to its axes may be expressed thus:

$$y^2 + (1 - e^2)\ x^2 = (1 - e^2)\ A^2,$$

$$\therefore y^2 = (1 - e^2)\ (A^2 - x^2).$$

(641.) If in a circle, P T (*fig.* 220.) be a tangent at P, and P M an ordinate to the diameter A A′, the right-angled triangles C M P and C P T being similar, we shall have C M to C P as C P to C T. But C P

being equal to C A, it follows that C M, C A, and C T, are in continued proportion.

(642.) The projection of A A′ and P M being a diameter of an ellipse and its ordinate, and the projection of P T being a tangent to the ellipse, and the projections of C M, C A, and C T, being proportional to those lines themselves, it follows if from any point, P, (*fig.* 221.) in an ellipse, a tangent P T be drawn, and from the same point an ordinate P M be drawn to the diameter C T, the lines C M, C A, and C T, will be in continued proportion. For these lines are the projections of the lines C M, C A, and C T (*fig.* 220.).

(643.) Hence a tangent may be drawn to an ellipse, from a point outside it. Let the given point be T, (*fig.* 221.) Find the centre C of the ellipse (632.),

fig. 220.

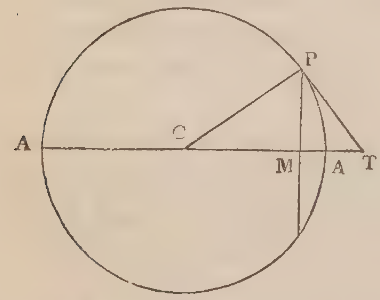
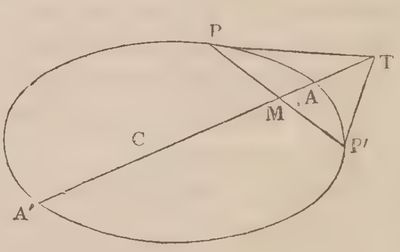


fig. 221.



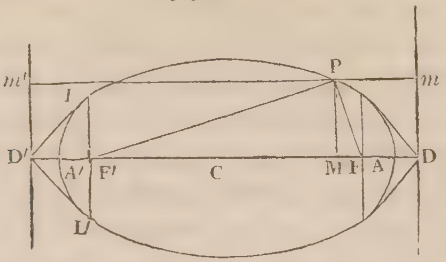
and draw T C. Find a third proportional to C T and C A, and take C M equal to this third proportional. Through M draw an ordinate to the diameter C A, or, what is the same, a line parallel to its conjugate diameter which may be found by (633.); and from the point P, where this ordinate meets the ellipse, draw P T. This ine will be a tangent to the ellipse at P (642.)

(644.) It is evident that tangents through P and P′, the extremities of the same ordinate, will meet the diameter, produced at the same point T; for the distance of this point from the centre C will be a third proportional to C M and C A, whichever of the points P, P′, the tangent be drawn from.

(645.) If an ordinate to the transverse axis be drawn through the focus F or F′, the tangent drawn through

its extremities (*fig.* 222.) will meet the axis at a point

fig. 222.



D or D′, whose distance from the centre is a third proportional to C F and C A. So that

C F : C A : C D.

(646.) Lines drawn perpendicular to the transverse axis, through the points D, D′, are called *directrices* of the ellipse.

(647.) It is a property of the directrices, that the distance P m of any point in the ellipse from either of them, is everywhere proportional to the distance P F of the same point from the focus.

This admits of being easily proved by the notation of algebra:—

Since $\mathrm{F\,C}=c$, $\mathrm{A\,C}=\mathrm{A}$, and $\mathrm{C\,M}=x$, we shall have $\mathrm{C\,D}=\frac{\mathrm{A}^2}{c}=\frac{\mathrm{A}}{e}$, and therefore

$$\mathrm{P}\,m=\mathrm{D\,M}=\frac{\mathrm{A}}{e}-x=\frac{\mathrm{A}-ex}{e}$$

But since $\mathrm{P\,M}=y$ and $\mathrm{M\,F}=c-x$, we shall have

$$\mathrm{P\,F}^2=y^2+(c-x)^2=y^2+(\mathrm{A}e-x)^2;$$

$$\text{but } y^2=(1-e^2)\,(\mathrm{A}^2-x^2),$$

$$\therefore\ \mathrm{P\,F}^2=(1-e^2)\,(\mathrm{A}^2-x^2)+(\mathrm{A}e-x)^2=(\mathrm{A}-ex)^2,$$

$$\therefore\ \mathrm{P\,F}=\mathrm{A}-ex.$$

Hence we have

$$\frac{\mathrm{PF}}{\mathrm{P}m}=\frac{(\mathrm{A}-ex)e}{\mathrm{A}-ex}=e$$

Hence, the distance of any point in the ellipse from the focus F, is to its distance from the directrix D m, as the number e is to 1, or as F C is to A C, that is, as

the distance of the focus from the centre is to the semi-transverse axis.

In like manner, it may be proved, that the distance Pm' of the point P from the other directrix is to its distance P F′ from the other focus in the same ratio.

(648.) The double ordinate L L′ (*fig.* 222.), to the transverse axis, which is drawn through the focus, is called the *principal parameter* of the ellipse.

Let F′ L$=p$. By the equation

$$A^2 y^2 + B^2 x^2 = A^2 B^2$$

we have $$A^2 p^2 + B^2 c^2 = A^2 B^2$$

and since $c^2 = A^2 - B^2$ we shall have

$$A^2 p^2 = B^4$$

$$\therefore\ p = \frac{B^2}{A}$$

that is, the ordinate p is a third proportional to the semi-transverse and semi-conjugate axis, and therefore the parameter L L′ or $2p$ is a third proportional to these two axes.

(649.) Since, $$\frac{p}{A} = \frac{B^2}{A^2}$$

The equation of the ellipse may be expressed thus:—

$$y^2 + \frac{p}{A} x^2 = p\,A$$

(650.) Since C M$=$A′ M $-$ A′ C, if A′ M$=x'$ we shall have, $$x = x' - A$$

$$\therefore\ y^2 + \frac{p}{A}(x' - A)^2 = p\,A$$

$$\therefore\ y^2 + \frac{p}{A} x'^2 = 2\,p\,x'$$

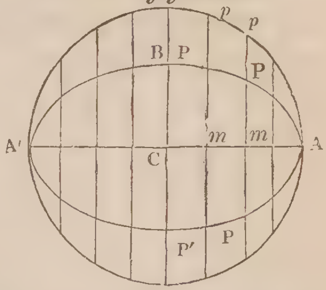
which is the equation of the ellipse when the abscissæ x' are taken from the vertex A′.

(651.) In the application of geometry in the arts, it is frequently necessary to trace a curve not by the continued motion of a pencil or stile, but by determining a number of separate points of the curve so near each other, that the intermediate parts may be filled up by giving the line the curvature indicated by the successive positions of the points, or, with still greater precision, by

determining the radius of the curvature of the curve corresponding to each of the points, as will be explained hereafter.

An ellipse may be described by determining a succession of points in it in several ways. If a circle be described on the transverse axis A A′ (*fig.* 223.), of the ellipse as a diameter, and A A′ be divided by a number of points *m*, at short distances from one another, and through these points perpendiculars *m p* to the axis be drawn, terminating in the circle. If these ordinates *mp*, to the diameter of the circle, be divided at P, so that *p m* may be every where to P *m*, the part cut off, as the transverse axis of the proposed ellipse to its conjugate axis, then the points P of division of the ordinate will be all placed on the ellipse, and points P′ may be similarly determined in the same ellipse below the axis.

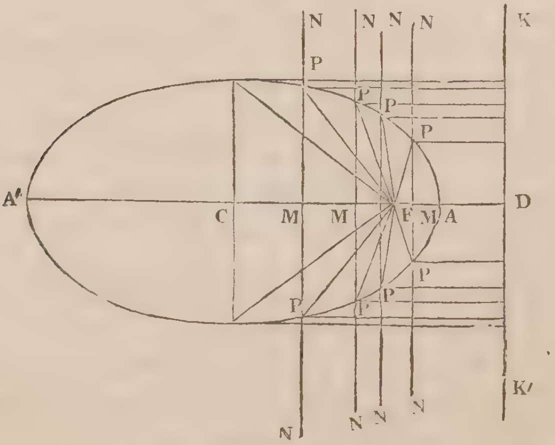
fig. 223.



The curve drawn through the points thus determined will be the ellipse required.

Otherwise thus :—Let C (*fig.* 224.), be the centre,

fig. 224.

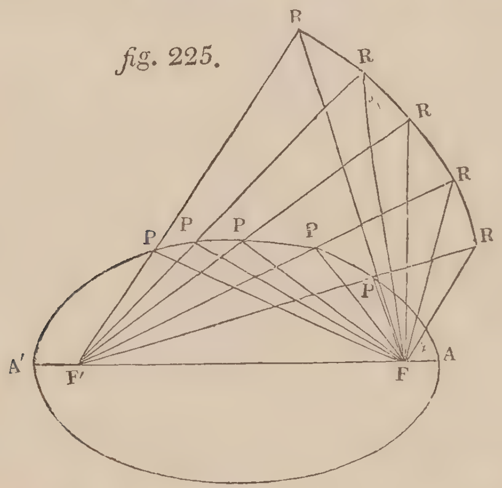


F the focus, and A the vertex of the required ellipse. From any points in the axis A A′ let perpendiculars,

M N, be drawn, and taking a third proportional to C F and C A, let it be C D, and through D draw K K′ perpendicular to C D. This line K K′ will be the directrix of the required ellipse. From F let a line F P be inflected on each of the perpendiculars M N, of such a length that it shall have to the corresponding distance M D the same ratio as C F has to C A. The points P will then be on the ellipse, in virtue of the property of the directrix explained in (647.)

Otherwise thus, let F and F′ (*fig.* 225.) be the foci of

fig. 225.

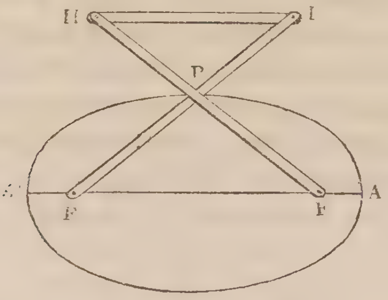


the required ellipse, and A A′ its transverse axis. With F′ as a centre and a radius equal to the transverse axis, let the arc of a circle be described, and from F′ draw any number of radii F′ R to this arc. From F draw the lines F R to the points where these radii meet the arc of the circle, and from F draw the lines F P making with the lines F′ R angles equal to the angles F R F′. The points P will then be on the ellipse, for since the angle F R P is equal to the angle R F P, the side F P will be equal to the side R P, and therefore the sum of the sides F′ P and F P will be equal to F′ R or to A′ A, that is, to the transverse axis of the ellipse; the points P will therefore be in the ellipse.

(652.) Besides the method of describing an ellipse by a continuous motion, explained in (581.), that curve may also be described by a continuous motion in the

following manner:—let A A′ (*fig.* 226.) be the transverse axis of the proposed ellipse, and let F and F′ be

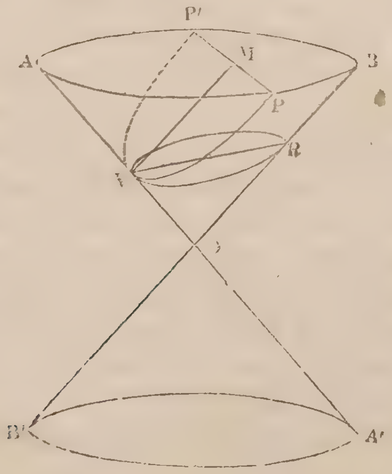
fig. 226.



its foci. Let F H and F′ I be two rulers attached by pivots to the foci, each equal in length to the transverse axis, and let H I be a third ruler equal to F F′ the distance between the foci. Let a slit be formed along the ruler F H, and another along the ruler F′ I, and let a pencil be inserted at P, the point where these two slits cross each other, so that, passing through the two slits, it may press on paper under the system of rulers; let the rulers be moved so as to turn round the points F and F′ as centres, and the pencil, following the point of intersection of the slits, will trace the ellipse.

(653.) If a cone A O B (*fig.* 227.) be intersected by

fig. 227.



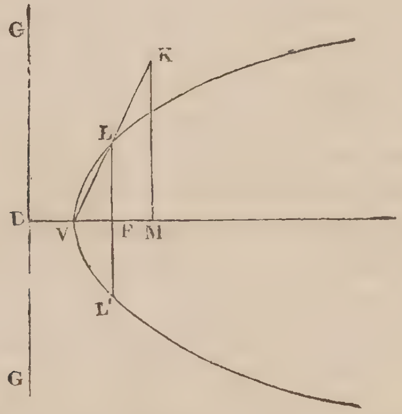
T

a plane P V P′, parallel to its side B B′, the curve which will be formed by the section is called a *parabola*. If the plane A B O be perpendicular to the cutting plane, their line of intersection V M will be the axis of the parabola, and all cords, such as P P′, drawn perpendicular to this and terminated in the curve, will be bisected by it.

(654.) It is evident that the intersecting plane cannot meet the opposite cone A′ O B′, being parallel to B B′, and therefore no part of the parabola can lie below the vertex V; and, as the cutting plane cannot meet the line O B above O, the branches V P and V P′ of the parabola must go on diverging with the divergence of the conical surface, and will thus extend without limit in that direction.

(655.) Let M (*fig.* 228.) be any point taken on the

fig. 228.



axis of a parabola. On a perpendicular to the axis, through M, take a distance M K equal to twice M V, and draw K V; from the point L, where K V meets the curve, draw L F perpendicular to the axis. The point F is called the *focus* of the parabola.

(656.) Take V D equal to V F, and through D draw D G perpendicular to D M; the line D G is called the *directrix* of the parabola.

(657.) The ordinate F L to the axis through the focus is equal to twice the distance F V of the focus from the

vertex, and therefore equal to F D, the distance of the focus from the directrix; for K M is to M V as L F is to F V; but K M was taken equal to twice M V, and therefore L F is twice F V.

(658.) If, while the length of the parameter L L′ (*fig.* 222.) of an ellipse, and the position of the vertex A′, and the axis A′ A is preserved, the centre C be supposed to recede indefinitely, so that the length of the axis A′ A shall increase without limit, the form of the ellipse will approach to that of a parabola, and will approximate to it without limit. This is what would take place if the plane, which passes through V (*fig.* 227.), and intersects the conical surface A O B, should first intersect that surface in a direction V R, meeting the side O B, and making an ellipse by its section, and then turning on the point V, the angle R V O, made by the cutting plane with V O, should be gradually increased; the point R would gradually recede from O, and the ellipse would be constantly elongated, while the angle R V M, under its plane and that of the parabola, would be constantly diminished; the cutting plane would at length become parallel to O B; the point R would be removed to an infinite distance; or, in other words, the transverse axis, V R of the ellipse, would become infinite, and the ellipse would become a parabola.

(659.) In the equation

$$y^2+\frac{p}{A}x'^2=2px'$$

if we suppose p to be of definite magnitude, and A to become infinite, $\frac{p}{A}=0$; therefore, the equation would become

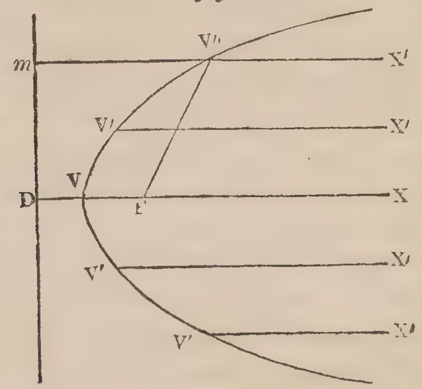
$$y^2=2px'$$

which is the equation of the parabola, and which, being translated into ordinary language, is a statement of the proposition, that the square of the ordinate y to the axis of a parabola, is equal to the rectangle under the distance x' of that ordinate from the vertex of the curve and

the parameter $2p$, or double ordinate LL', through the focus.

(660.) Since the centre, or common point of intersection of the diameters of the ellipse recedes to an infinite distance when the ellipse becomes a parabola, these diameters therefore become parallel to each other and to the axis. Hence all lines in a parabola, such as $V' X'$ (*fig.* 229.) parallel to the axis $V X$, are diameters.

fig. 229.



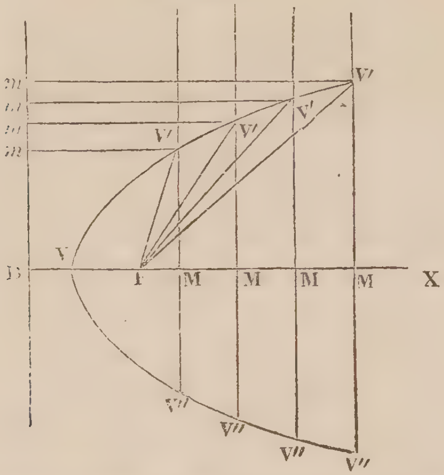
(661.) Each of the lines $V' X'$ will bisect a system of chords parallel to a tangent through V', which chords will be ordinates to these diameters respectively.

(662.) The distance $F V'$ of any point V' in a parabola from the focus, is equal to its distance $V' m$ from the directrix.

In the ellipse, the ratio of these distances was shown to be that of the distances of the centre from the vertex and the focus. When the ellipse becomes a parabola, these two distances become infinite, while their difference, or the distance $V F$, remains finite. Their ratio, therefore, becomes a ratio of equality, and the line $D' m'$ (*fig.* 222.) becomes the line $D m$ (*fig.* 229.), the distance $D F$ being now bisected at V, instead of being divided as in *fig.* 222. in the ratio of c to A. In like manner, the ratio of the distance $V' F$ of any point V' on the curve from the focus, to its distance $V' m$ from the directrix, instead of being that of c to A is a ratio of equality.

(663.) The property just explained supplies a method of constructing or drawing a parabola by a series of points. Let F (*fig.* 230.) be the focus, and V the vertex of

*fig*2. 30.



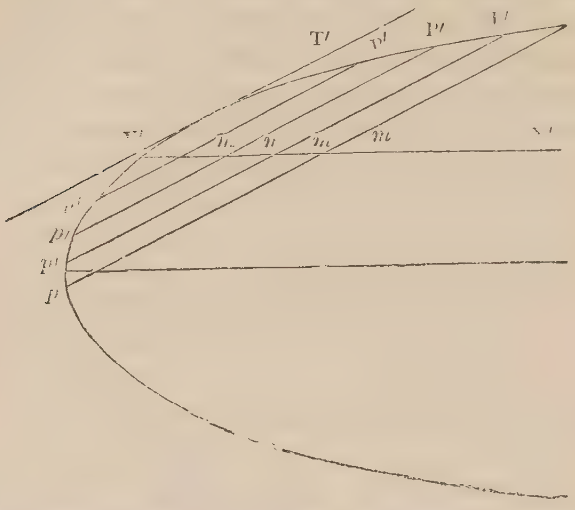
the proposed parabola. The line V F produced will be its axis; and if V D be taken equal to V F, and D *m* be drawn perpendicular to V X, D *m* will be the directrix. Taking any points M on the axis, let perpendiculars be drawn through them, and from the point F let lines F V′ be inflected on each perpendicular equal to M D, the distance of that perpendicular from D. The points V, thus determined, will be points of the parabola; and if points V″ be taken at equal distances below the axis on the perpendicular, they will be the corresponding points on the lower branch of the curve.

These points on each branch may thus be formed as numerously and as close together as may be desired, and a curve drawn through them will therefore be the parabola.

(664.) Since the diameters of the ellipse preserve their properties as the centre recedes from the vertex, their ordinates will still be parallel to tangents through their vertices; hence, every diameter of a parabola will bisect a system of chords parallel to a tangent to the curve through its extremity, as represented in *fig.*231., where V′ X′ is a diameter, V′ T′ a tangent through its

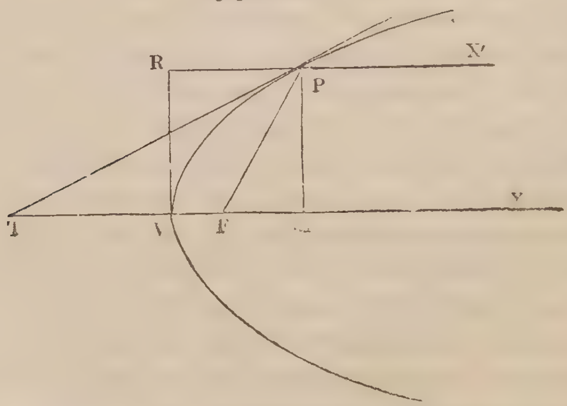
extremity, and P′ p′ are double ordinates bisected by the diameter V′ X′ at *m*.

fig. 231.



(665.) While the centre C (*fig.* 222.) of the ellipse recedes from its vertex, the focus F also recedes from A′; and when the ellipse becomes a parabola, the further focus will be removed to an infinite distance. If, therefore, P (*fig.* 232.) be a point in the parabola, a line

fig. 232.



drawn from P to the remote focus will be parallel to the axis V X, since its intersection with that axis will be at an infinite distance; X′ P and F P may therefore

be regarded as the ultimate position, which lines, drawn from P to the foci of the ellipse, assume, when the ellipse becomes a parabola. Since these lines, from the foci of an ellipse, are inclined at equal angles to a tangent, the lines which correspond to these in the parabola will have a like property; and since these lines are the lines drawn from the point of contact P to the focus and the diameter P X drawn from the same point, these lines will form equal angles with the tangent T T′.

(666.) To draw a tangent, therefore, to a given point P in a parabola, whose axis and focus are given: from the focus draw F P and take F T equal to F P, and draw T P. This line T P will be a tangent to the parabola at P; for, since F T and F P are equal, the angle T is equal to the angle F P T; but since P X is parallel to T X, the angle T′ P X′ is equal to the angle T: therefore the angles F P T and X′ P T′ are equal.

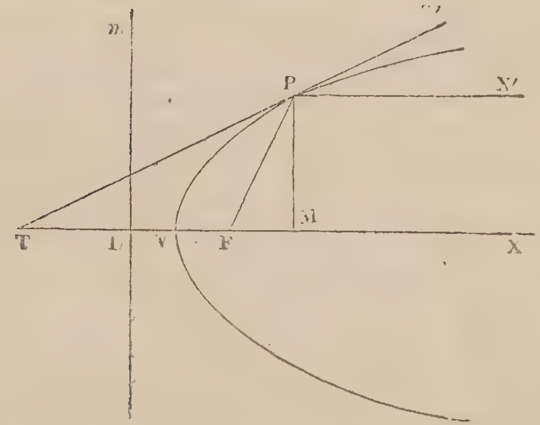
(667.) If X′ P be a ray of light, heat, sound, or any other physical principle which obeys the common law of reflection, and the curve at P have the property of reflection, the ray X′ P will be reflected from P to F, and the same will be true of all rays which have directions parallel to the axis X V. If the curve revolve on its axis X V, so as to produce a paraboloid of revolution, the surface of such a figure will have the property of reflecting, to its focus F, all rays which strike it in directions parallel to its axis; and, on the other hand, if a luminous object be placed in F, the focus of such a surface, the rays diverging from it, will be reflected by the surface in parallel lines. The reflectors of lighthouses and beacons are sometimes constructed of this form: a copper surface being produced in the shape of a paraboloid of revolution, and highly plated and burnished, the lamp being placed in the focus, a cylinder of parallel rays will be reflected from the surface, and thrown across the horizon in the direction in which the light is intended to be seen.

If such a reflector had a fixed position, the beam of light reflected from it would only be visible to ships in

one particular direction: to remedy this, the reflector is placed upon a vertical axis on which it is made to revolve, and as it revolves the beam of light, reflected from it, sweeps the horizon in every direction round the axis of revolution, so that the light becomes visible in each direction once in each revolution. As such lights are numerous on the same coast, and often placed at short distances asunder, the mariner is enabled to distinguish them one from another, and thereby to know the position of his ship at night, by observing the interval between the successive appearances of the light. The axis on which the mirrors revolve is regulated in its motion by clockwork, and the mirrors of different lighthouses are made to revolve in different intervals of time.

(668.) If from the point of contact P (*fig.* 233.), an

fig. 233.



ordinate P M be drawn, the distance M T will be bisected by V, the vertex of the parabola; for if D *m* be the directrix, it has been already proved that F P is equal to M D; or, since V D is equal to V F, F P is equal to M V together with V F; but F P is also equal to F T or to V T together with V F, therefore M V is equal to V T.

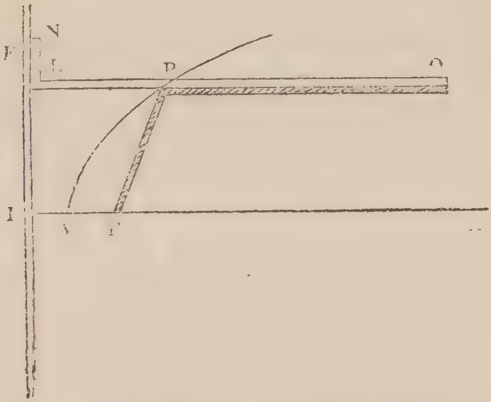
(669.) Hence a tangent may be drawn to a parabola from any point T, in the production of its axis; for,

take V M equal to V T, and through M draw a perpendicular M P, the line P T will then be the tangent required.

(670.) When a diameter is given, the direction of its ordinates may be found, for they are parallel to a tangent drawn through its extremity.

(671.) To describe a parabola by a continued motion, let D K (*fig.* 234.) be a straight ruler placed upon the directrix of the parabola, and let D V X be the axis of the parabola; let L O be another straight ruler having a short leg L N at right angles to it, placed against D K, so that by sliding L N upon D K, L O shall always be perpendicular to D K. Let F be the focus of the proposed parabola, and let a flexible thread, having one end fixed to F, be stretched to such a point P on the ruler

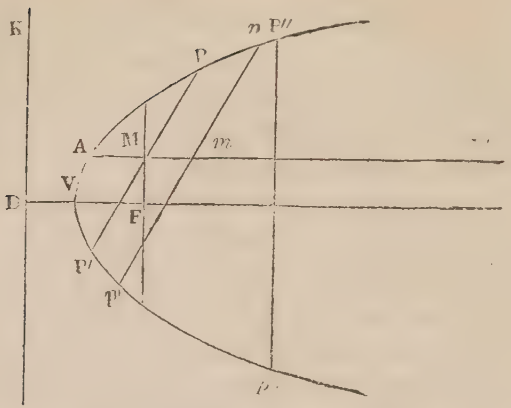
fig. 234.



L O, that F P shall be equal to P L, and let it be then extended along the ruler, and attached to its extremity O. Let a pencil be placed in the loop of the thread at P, holding the thread stretched tight, and so that the pencil shall be constantly pressed against the ruler. If L N be then moved so as to slide in either direction on D K, the pencil, still pressing on the ruler, and keeping the cord extended, will describe the required parabola V P.

(672.) In a given parabola to determine the axis, focus, and directrix, draw any two parallel cords P P′, and $p\,p'$ (*fig.* 235.), bisect them, and through their

fig. 235.



points of bisection M, *m*, draw a line A X′. This line will be a diameter, of which P P′, *p p′* are ordinates, and therefore A X′ will be parallel to the axis of the parabola. To find the axis, draw any line P″ *p″* at right angles to A X′, and bisect it, and through its point of bisection M, draw M V at right angles to it. This line M V will be the axis of the parabola. The axis being known, the focus F may be found by the method explained in (655.); and if V D be taken, equal to V F, and D K be drawn perpendicular to D M, D K will be the directrix.

(673.) To draw a diameter of a parabola which shall make a given angle with its ordinates.

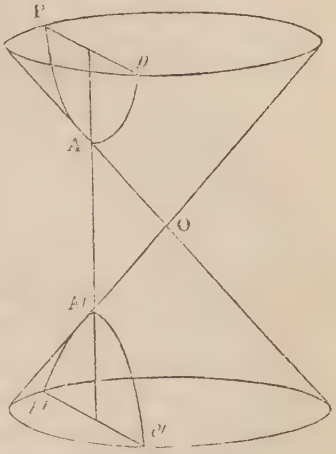
Let F be the focus and V X the axis (*fig.* 233.). Draw F P, making the angle P F X equal to twice the given angle, and take F T equal to F P, and draw T P, which will be a tangent to the parabola at P. From P draw P X′ parallel to V X, and P X′ will then be the diameter required; for, its ordinates being parallel to the tangent P T, will make with it an angle equal to the angle T; but since F P is equal to F T, the angle P F X will be equal to twice the angle T, which latter will therefore be equal to the given angle.

(674.) The area included by the ordinate P M (*fig.* 232.), the abscissa V M, and the parabolic arc P V, is shown by the method of quadratures in the higher analysis to be two thirds of the rectangle M R,

that is, two thirds of the rectangle under the ordinate and the abscissa.

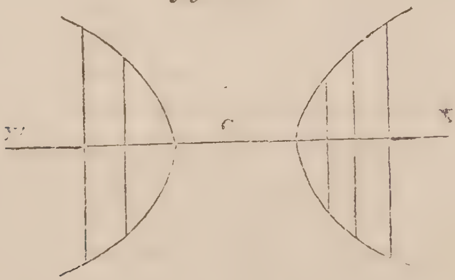
(675.) If a cone (*fig.* 236.) be cut by a plane so placed as to intersect those parts of the conical surface A A′, which are on opposite sides of the vertex, the section, consisting of two curved branches P A p and P′ A′ p', having their convexities turned towards each other, will be the curve called an *hyperbola*.

fig. 236.



(676.) The opposite branches of an hyperbola being formed by the section of the same plane, with perfectly similar conical surfaces, will be curves precisely similar to each other, and their axis X′ X (*fig.* 237.) will divide them symmetrically, bisecting the ordinates to it as in the ellipse.

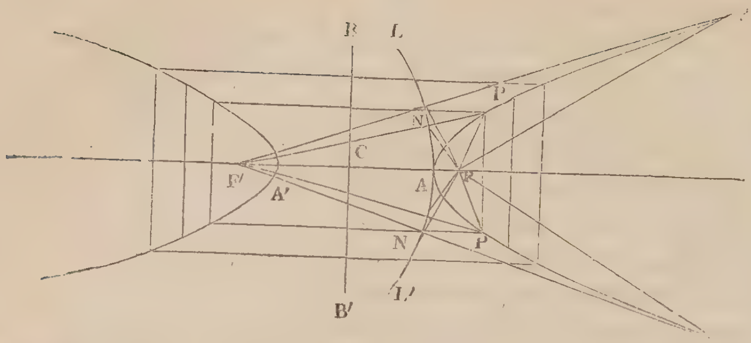
fig. 237.



(677.) It was shown, that if any number of triangles be constructed on the same base, of which the *sum* of the sides shall be equal, their vertices will lie in an ellipse, whose transverse axis is equal to the sum of their sides. The hyperbola possesses a property analogous to this; for, if several triangles stand on the same base, of which the *difference* of the sides shall be the same, their vertices will lie in an hyperbola, of which the difference of the sides shall be the transverse axis.

Let F and F′ (*fig.* 238.) be two fixed points, and let

fig. 238.



A be the vertex of an hyperbola, of which F and F′ will be the foci, and A′ the other vertex, the distance F′ A′ being taken equal to the distance F A. With F′ as centre and a radius equal to F′ A, let the arc of a circle L L′ be described. From F′ draw any radius F′ N of this circle, and draw F N. Produce F′ N, and from F draw a line, making with F N an angle equal to that which the production of F′ N makes with it. Let these lines intersect at P. The point P will then be the vertex of a triangle, whose base is F F′, and the difference of whose sides is F′ N. If several points be determined in the same manner they will be points of an hyperbola, whose foci are F and F′, and whose vertex is at A. Any number of points being thus determined as nearly together as is desired, the curve which shall pass through them will be an hyperbola. The centre of this hyperbola will be at C, the point of bisection of A A′. By a like process, the branch of the curve on the opposite side, passing through A′ as its vertex, may be determined.

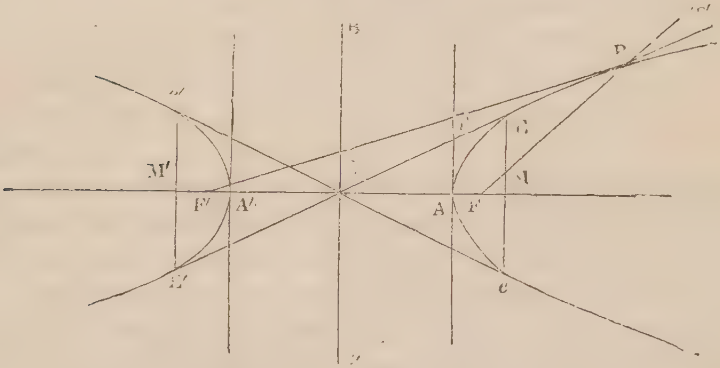
(678.) From the manner of determining the branches of the curve just explained, it will be evident that the parts of it contained within the four right angles, formed by the lines A A′ and B B′, are perfectly equal and similar, and symmetrically placed with regard to these lines; so that if the paper on which the curves are drawn be doubled over, making a fold along the line A A′, the lower branches would fall upon the upper; and if the paper were doubled over, making a fold along the line B B′, the right branches of the curve would fall upon the

left. Thus, it appears that all lines terminated in either branch of the curve, parallel to B B′, are bisected by the production of A A′, and all lines terminated in the opposite branches, and parallel to A A′, are bisected by B B′.

(679.) The lines A A′ and B B′ are therefore *conjugate diameters* of the curve, the line A A′ being called the *transverse axis*, and the line B B′ the *conjugate axis*.

(680.) The leading properties of an hyperbola have a close analogy to those of an ellipse. From the symmetry and equality of the four branches of the curve, contained in the four right angles formed by its axes, it follows, that all right lines, such as E E′ (*fig.* 239.)

fig. 239.



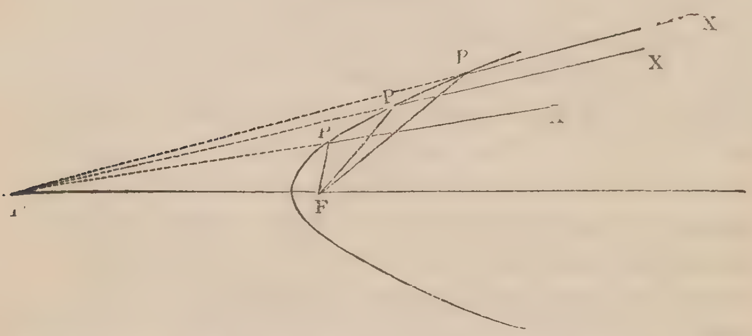
drawn through the centre C and terminated in opposite branches of the curve, will meet these branches at corresponding points, and that the centre C will be their common point of bisection. The ordinates to the axis E M and E′ M′, through the extremities of the same diameter, are equal; and it is evident, that another diameter *e e*′, through the other extremities of these ordinates to the axis, will be equal to the former, and equally inclined to the axis, so that M M′ will bisect the angles E C *e* and E′ C *e*′.

(681.) Lines drawn from the foci of an hyperbola to a point P in it, make equal angles with the tangent at that point, in the same manner as was shown to be the case in the ellipse; but in the case of the hyperbola, the

tangent lies between these lines F P and F′ P, bisecting the angle under them, whereas, in the ellipse, it lay outside them. Tangents to the hyperbola at the vertices A and A′ are perpendicular to the axis, and therefore parallel to its ordinates; and, in like manner, tangents to the curve, through the extremities of any diameter, are parallel to the ordinates to that diameter.

(682.) If F (*fig.* 240.) be the focus of an hyperbola,

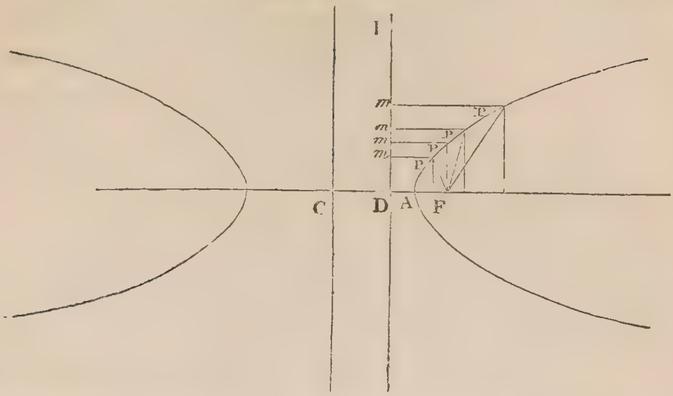
fig. 240.



it appears from what has been stated, that rays of light, or heat, or any other principle which obeys the law of reflection, will, if they diverge from F, and are reflected from the curve at P after reflection, follow directions diverging from the other focus F′; and, on the other hand, if rays X P, converging towards F′, be reflected by the curve, they will after reflection converge towards the other focus F. Hence, if an hyperboloid of revolution be formed by the revolution of an hyperbola on its transverse axis, and such a surface be endowed with the property of reflection, rays converging to, or diverging from, one focus, may be made to converge to, or diverge from, the other focus.

(683.) If C (*fig.* 241.) be the centre, A the vertex, and F the focus of an hyperbola, take C D a third proportional to C F and C A, and through D draw D K perpendicular to C A; the line D K will be the directrix of the hyperbola, and is distinguished by properties analogous to the directrix of the ellipse (647.). Let P be any points on the hyperbola, from which let the lines P m be drawn parallel to the axis, and there-

fig. 241.



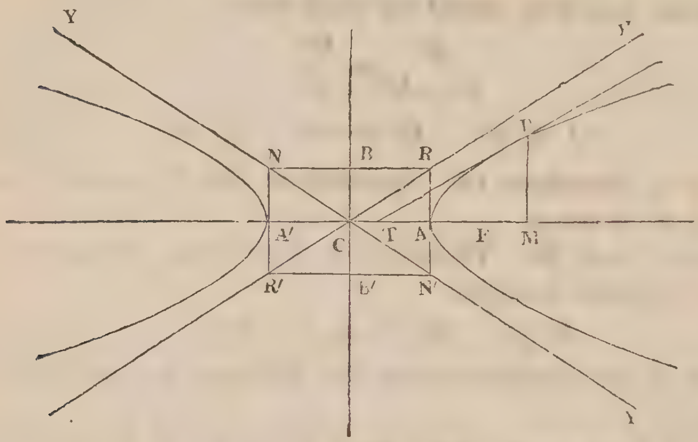
fore perpendicular to the directrix, the ratio of each of the lines F P to P m will be the same as the ratio of F A to A D, or, what is the same, that of F C to A C. Thus the line D K is distinguished by the property of having the distances of all points in the curve from it proportional to the distances of those points from the focus.

(684.) From this property, a method of determining the curve, by a series of points, follows, perfectly similar to the method of determining the ellipse by points in reference to its directrix explained in (651.). It is only necessary to draw any number of lines perpendicular to the axis, and from the focus to inflect on each of them a line which shall bear to their distances respectively the same ratio as the distance of the focus from the centre bears to the semi-transverse axis.

(685.) If T P (*fig.* 242.) be a tangent to the hyperbola at P, and P M be an ordinate to the axis at the same point, then, by a property analogous to that of the ellipse, already explained (642.), we shall have C T to C A as C A to C M; and, therefore, the rectangle under C T and C M is always equal to the square of the semi-axis; and, therefore, as C M increases C T must diminish. But as the hyperbola extends indefinitely from A in the direction A P, consisting of an infinite branch, the distance C M increases without limit, as the point of contact P recedes, and therefore the distance C T at the same time diminishes without limit. If the point

of contact P be removed to an infinite distance, then C M becomes infinite, and C T vanishes. Hence it

fig. 242.



appears that the tangent to the curve always intersects the axis between C and A, but that the farther the point of contact is removed from the vertex, the nearer the tangent approaches to the centre; and that the curve has a constant tendency to coincide with a certain line passing through the centre, although it never can actually coincide with such a line, since that would involve the condition of its being at an infinite distance from the centre.

(686.) It was shown among the properties of the ellipse, that the square of the ordinate P M to the axis always bears the same ratio to the rectangle under the distances between that ordinate and the extremities A A′ of the axis; this ratio being that of the square of the semi-conjugate to the square of the semi-transverse axis. In like manner the square of P M (*fig.* 242.) bears to the rectangle under M A and M A′, the same ratio, wherever the point P is taken. Let a distance C B be taken on the conjugate axis, such that the square of C B shall bear to the square of C A, the same ratio as the square of any ordinate bears to the rectangle under the corresponding segments. This distance C B is considered as the length of the semi-conjugate axis, although it does not, as in the ellipse, meet the curve at B.

The properties of the hyperbola may be expressed

by the same notation as was used to express the properties of the ellipse in reference to its axes.

Let $A = CA$, $B = CB$, $y = PM$, and $x = CM$. By what has been stated we shall have

$$\frac{y^2}{x^2 - A^2} = \frac{B^2}{A^2},$$

$$\therefore A^2 y^2 - B^2 x^2 = -A^2 B^2,$$

which is therefore the equation of the hyperbola referred to its axes.

Let $x' = AM$. Therefore $A + x' = x$. Hence the above equation becomes

$$A^2 y^2 - B^2 x'^2 = 2 A B^2 x',$$

which is the equation when the abscissæ are measured from A.

If $FC = c$, we shall have $c^2 = A^2 + B^2$; and if $e = \frac{c}{A}$ we shall have

$$A^2 y^2 - A^2 (e^2 - 1) x'^2 = 2 A^3 (e^2 - 1) x',$$

$$\text{or } y^2 - (e^2 - 1) x'^2 = 2 A (e^2 - 1) x'.$$

(687.) Since the distance C T (*fig.* 242.) diminishes without limit as the point of contact P recedes, it is evident that when the distance of P is infinite, the tangent would pass through the centre C. Its ultimate direction, or more strictly the direction which limits its position as the point of contact recedes without limit, may be determined by finding the value to which the ratio of P M to M T, or of P M to M C tends when they both become infinite. This will be readily obtained from the equation

$$A^2 y^2 - B^2 x^2 = -A^2 B^2.$$

Dividing all the terms by $A^2 x^2$ we obtain

$$\frac{y^2}{x^2} - \frac{B^2}{A^2} = -\frac{B^2}{x^2},$$

$$\text{or } \frac{y^2}{x^2} = \frac{B^2}{A^2} - \frac{B^2}{x^2}.$$

Now, when x is infinite, $\frac{B^2}{x^2} = 0$, and therefore

$$\frac{y^2}{x^2} = \frac{B^2}{A^2}, \text{ or } \frac{y}{x} = \pm \frac{B}{A}.$$

The limiting values of $\frac{PM}{CM}$ are, therefore, $\pm \frac{B}{A} = \frac{CB}{CA}$.

If through B and B′ parallels N R and N′ R′ to A A′ be drawn, and through A and A′ parallels N′ R and N R′ to B B′ be drawn, the diagonals R′ R and N′ N of this rectangle will be the positions to which the curve ultimately tends as it recedes from its centre.

For B A is to A C as any perpendicular drawn from a point in C R produced is to the distance of such perpendicular from C; and as this is the same ratio as the limiting ratio of P M to C M, it is evident that P M ultimately tends to equality with such perpendicular as C M is increased.

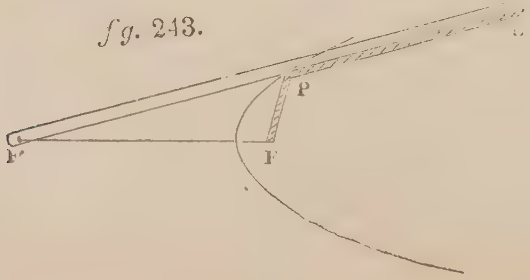
The line C N′ produced has the same relation to the lower branch of the hyperbola.

The lines C Y thus determined are called asymptotes. An asymptote in general is a tangent drawn to a point of the curve at an infinite distance, or, more strictly, it is the limit of the position of the tangent, the distance of the point of contact being supposed to be continually and indefinitely increased.

(688.) Hence it is apparent, that the curve approaches its asymptote continually, the distance between them decreasing without limit, but never vanishing.

(689.) An hyperbola may be described by a continuous motion in the following manner:—To the focus F′ (*fig.* 243.) let a straight ruler F′ L be attached by a

fig. 243.



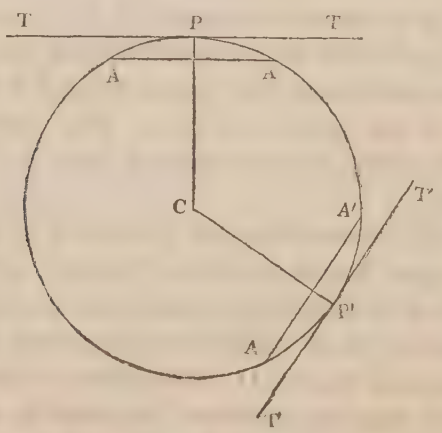
pivot, and to the other focus F let a flexible thread be attached by a pin, and let this thread be brought into contact with the ruler at P, and finally attached to its extremity at L. Let a pencil be looped in the thread at P, and held so that the thread shall be extended and the pencil pressed against the ruler. Let the ruler be thus turned slowly round the pivot F′, and the pencil will describe an hyperbola whose transverse axis will be the difference between F P and F′ P.

CHAP. XXI.

OF THE CURVATURE OF CURVES.

(690.) The degree of curvature or flexure of a curve is estimated by the rapidity with which the point describing the curve departs from the tangent as it leaves the point of contact in either direction. A circle differs from all other curves whatever in having a perfectly uniform curvature throughout its whole circumference. If a tangent be drawn to any point in a circle, the arc of the circle, extending on either side of the point of contact, will be situated in exactly the same manner as an arc of the same circle would be with respect to a tangent at any other point. Thus if P (*fig.* 244.) be the point of

fig. 244.

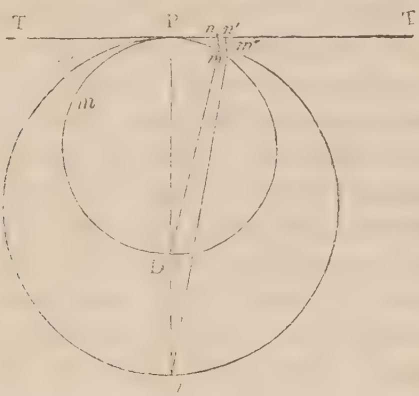


contact of a tangent P T, and P′ be the point of contact of another tangent P′ T′, the arc P A on either side at P will be placed similarly to the arc P′ A′ on either side of P′. That this will necessarily be the case will be evident by considering that, if a segment A′ A′ be

cut off by a chord, and the arc cut off be removed, and so placed that the point P′ shall lie upon P, and the line P′ T′ on the line P T, the arc P′ A′ will lie upon the arc P A, and the same will be the case to whatever points in the circle the tangents may be drawn.

But if two circles have different magnitudes, they will then have different curvatures. Let P D and P D′ (*fig.* 245.) be the diameters of two such circles, to which

fig. 245.



P T shall be a common tangent at P. It is evident that the lesser circle will be contained within the greater, and that its circumference will depart from P T more rapidly than that of the greater circle. The curvature, therefore, of the lesser will be greater than the curvature of the greater.

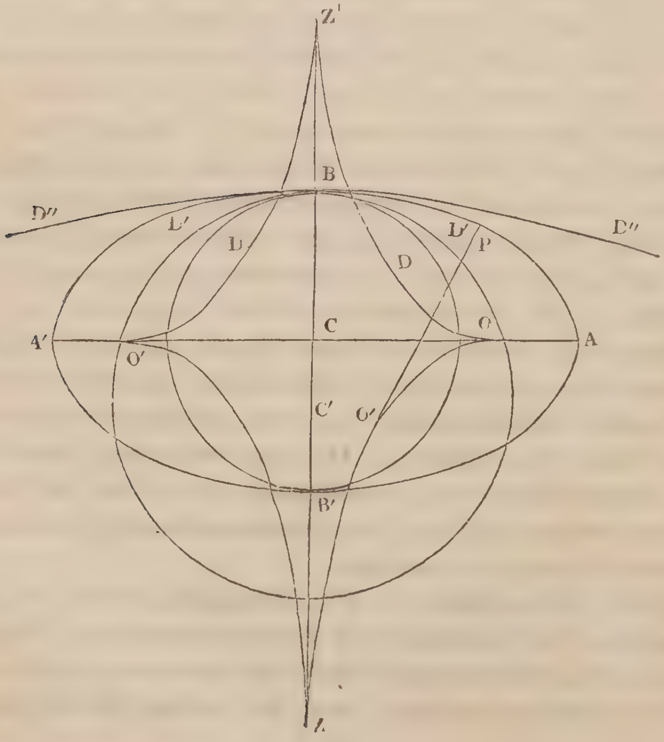
(691) If the curvature be measured by the departure of arcs of equal length from the tangent, let m, m' be the extremities of two such arcs, and let $m\,n$ and $m'\,n'$ be the lines measuring their respective departures from the tangent. Draw m D and m' D′; also draw the chords m P′ and m' P′, which may be considered to coincide with the arcs, the latter being very small. It will be easy to show from the common properties of the circle, that the rectangle under D′ P and $m'\,n'$ will be equal to the square of the arc P m', and the rectangle under D P and $m\,n$ will be equal to the square of the arc P m; but since

these arcs are equal, the rectangles under the diameters and the departures are equal; that is to say, in circles of different diameters the departures of equal arcs from their tangents are inversely as the diameters, therefore these diameters are inversely as the curvatures of the circles respectively.

(692.) The curvature of a circle being thus uniform, and the curvature of different circles being subject to unlimited variation by the increase or diminution of their diameters, the circle becomes the measure of the curvature of all other curves.

(693.) It will be evident, on inspection, that the curvature of an ellipse varies, gradually increasing from the extremities of its conjugate axis to the extremities of its transverse axis; the circle described with its conjugate axis as diameter, lies entirely within the ellipse, touching it at the points B B′ (*fig.* 246.); and since this circle

fig. 246.



departs more rapidly from the common tangent to it and the ellipse at B, than the ellipse does, the curvature of the circle is greater than the curvature of the ellipse at B. If the arc of a circle be described, having its centre on B B′ produced through B′, the radius may be taken of such a magnitude that the arc B D″ shall lie above the ellipse, and therefore between the tangent and the ellipse. Such a circle would therefore have a less curvature than the ellipse at B.

If the centre C′ of a circle passing through B be conceived to move downwards on the line B B′, the circle being at first under the ellipse on each side of the point B, would gradually approach it as the radius would be increased. The centre would at length reach a point on the axis B B′, or on that line produced, such that for all centres below it the circle on either side of B would lie above the ellipse, and for all centres above it, the circle would lie below the ellipse. It is evident, therefore, that all circles having their centres above this point would have a greater curvature than the ellipse, and all circles having their centres below it would have a less curvature. The circle, therefore, whose centre lies between the centres of those which pass above the ellipse on either side of B, and those which pass below it, comes nearer to the curvature of the ellipse than any other circle.

(694.) Such a circle is called the *osculating circle* of the ellipse, or the *circle of curvature* at the point B.

(695.) The investigation of the magnitude of the radius of the circle of curvature to any point in a curve requires the application of principles of analysis, higher and more difficult than can with propriety be introduced into this volume. We can only state, therefore, the magnitude of the osculating circle for particular curves, without giving any demonstration by which its magnitude may be obtained.

(696.) The radius of the osculating circle of the ellipse at either extremity of the transverse axis, is equal to a third proportional to the semi-transverse axis and the semi-conjugate axis; and the radius of curvature,

at the extremities of the semi-conjugate axis, is equal to a third proportional to the semi-conjugate axis and the semi-transverse axis. From the extremity A of the semi-transverse axis to the extremity B of the semi-conjugate axis, the radius of curvature gradually in- reases, its limiting magnitudes being those just stated.

(697.) To determine the radius of curvature for any point in the ellipse between A and B, let the semi-conjugate diameter to that which passes through the point be found, and let its cube be divided by the rectangle under the semi-axes. The quotient will be the radius of curvature corresponding to the given point.

(698.) A line drawn from the point of contact of a tangent, perpendicular to the tangent, is called a *normal* of the curve.

(699.) Since a line drawn perpendicular to the tangent to a circle, at the point of contact, must pass through the centre of the circle, it is evident that the centre of the circle of curvature must always lie upon the normal to the curve.

(700.) Since lines drawn from the foci of an ellipse are equally inclined to the tangent, they will also be equally inclined to the normal. The normal will, therefore, bisect the angle formed by lines from the foci to any point in the ellipse.

(701.) If the centres of the circles of curvature for all the points of the elliptical quadrant be determined, by taking upon the several normals distances equal to the radii of curvature, these centres will be found to be placed on a curve touching the transverse axis at a certain point O, and the conjugate axis at Z, the convexity of this curve being turned towards the centre C. The radius of curvature corresponding to any point P in the elliptical quadrant, will be a tangent to this curve at a certain point O′, P O′ being the radius of curvature corresponding to such point P.

(702.) Let a flexible thread be supposed to have one extremity fastened to Z and wrapped upon the curve

Z O′ O, and the other extremity be brought to A, the thread being unwound and at the same time kept extended, its extremity at A will move over the quadrant of the ellipse A B, and the part of the thread unwound from the curve at any point P will be the radius of curvature for that point.

(703.) The curve O Z on which the centres of curvature of any other curve A B are placed, is called the *involute* of that other curve. Thus, in the present case, the curve O O′ Z is the involute of the ellipse.

(704.) Since the curvature of the ellipse undergoes the same changes throughout each quadrant, the involute of B A′ is a curve O′ Z equal and similar to O Z lying in the angle A′ C Z, and in like manner the involutes of the elliptic quadrants A B′ and A′ B′ are similar curves O Z′ and O′ Z′ lying above the axis A A′.

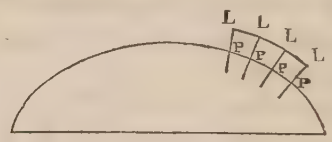
(705.) When a convenient practical method of describing a curve by one continuous motion of a pencil cannot be found, the curve may be determined with sufficient accuracy for all practical purposes by finding the centres of the circles of curvature for points in it separated by short intervals; arcs of the circles of curvature being described and extended through these intervals will give a line formed of a series of circular arcs differing so little from the curve sought, that they may be taken as representing it for any practical purpose.

(706.) If a thread having a pencil attached to it be wound upon a curve, the pencil as it is unwound, the thread being constantly extended, will describe a curve, the centres of curvature of which will lie upon the curve from which the thread is unwound; the curve described by the pencil is in this case called the *evolute* of the curve from which the thread is unwound.

(707.) In the construction of arches the formation of the voussoirs or arch-stones depends on the determination of the normals and radii of curvature to the different points of the curve, according to which the arch is formed. Let P P (*fig.* 247.) be a part of the arch of the width of a single arch-stone; the faces P L form-

ing the sides of the stones must be so cut, that, when fixed in their places, these faces shall be normals to the curve,

fig. 247.



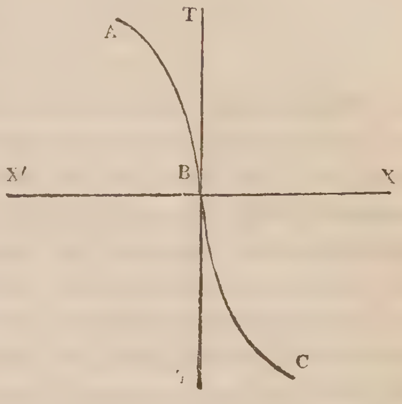
and the bottom or external face P P must form an arc of the circle of curvature to the curve at P. It will be apparent, therefore, that the correct form can only be given to such blocks by a due attention to those geometrical properties of the curve on which the determination of the normals and osculating circles depends.

(708.) As the radius of the osculating circle is an indication of the quantity of curvature, and as the variation of that radius shows the manner in which the flexure of a curve changes throughout any of its branches, so the position of the centre of the osculating circle, or, to use the language of analysis, the *sign* of the radius of curvature shows the direction of the curvature of the curve, that is, the side towards which the concavity is turned. According as the radius of curvature, algebraically considered, is positive or negative, the concavity is turned to the one side or the other. As a quantity which undergoes continuous variation can only change its sign, so as after being positive to become negative, or after being negative to become positive, either by vanishing or becoming infinite, it follows that, when the direction of the concavity changes, and therefore the radius of curvature undergoes a change of sign, it must pass through one or other of these states. If, on approaching the point where its sign changes it is in an increasing state, it will at the point where the sign changes become infinite, and the curvature there will become infinitely small, or the curve will at that place become very nearly a straight line. But if it be in a decreasing state as it approaches the point where it

changes its sign, it will then vanish at the point, and the curvature will become infinitely great.

(709.) Let A B (*fig.* 248.) be a part of a curve concave towards X′, and let the radius of curvature be supposed to increase as the curve approaches the point B, to which let X X′ be the normal. At B let the radius of curvature be infinite, and on passing below the normal, let its sign change. As the centre of curvature for B A was in the direction B X′, the centre of curvature for B C will be in the direction B X. Above B the concavity is therefore to the left, and below it to the right.

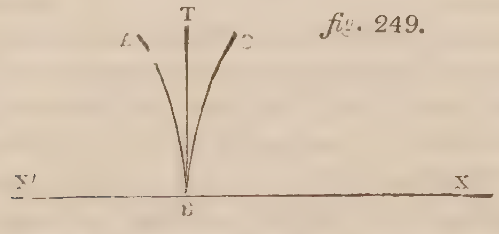
fig. 248.



(710.) Such a point as B is called a *point of contrary flexure*, or a *point of inflexion.*

(711.) If A B (*fig.* 249.) be a branch of a curve, of which the radius of curvature constantly diminishes

fig. 249.



in approaching B, and changes its sign in passing it, that radius will vanish at B, and the curvature at B will be infinitely great. After passing B, the curve will take the direction B C, X X′ being the common normal and B T the common tangent to the two branches of the curve.

(712.) Such a point as B is called a *Cusp*.

CHAP. XXII.

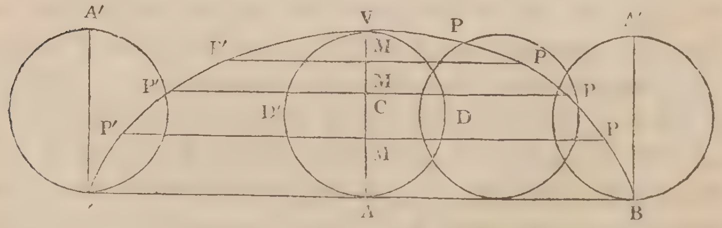
OF THE CYCLOID, THE CONCHOID, AND THE CATENARY.

(713.) THE diversity of curves which present themselves to the consideration of the geometer is infinite, and consequently the investigation of their individual properties can only be undertaken when those properties are called into play in the sciences or arts. General methods of investigation, by the aid of the language and principles of analysis, may be obtained by those who are willing to prosecute that branch of mathematics; meanwhile there are a few curves which have acquired a peculiar claim upon our attention, from the uses to which they are applied in physical and mechanical science. We shall devote this chapter to a brief exposition of their leading properties.

THE CYCLOID.

(714.) If a circle whose centre is C (*fig.* 250.), and whose

fig. 250.



radius is C A, touch a line B B′ at A, and while a pencil is attached at V and carried with the circle, the circle

itself is rolled along the line A B from A towards B, the pencil V will trace a curve V P B; and if, in like manner, it be rolled in the other direction from A towards B′, the pencil will trace an equal and similar curve V B′.

(715.) The curve B V B′ thus, defined, is called a *cycloid*, and is the curve which would be traced by a point situated on the edge of a carriage-wheel as that wheel is rolled in a straight direction on a level plane.

(716.) While the generating circle rolls from A to B, every point of its semi-circumference A D V applies itself to the line A B, and when the semicircle reaches the point B, the describing point V coincides with B, and the point A takes the position A′.

It is evident therefore that A B is equal to half the circumference A D V of the generating circle.

And in like manner, when the circle is rolled to B′ in the contrary direction, the point A takes the position A′, and the describing point V coincides with B′.

The distance A B′ is therefore rolled over by the semi-circumference A D′ V of the generating circle, and is therefore equal to that semi-circumference.

(717.) The line B B′ is called the *base* of the cycloid, and is equal to the circumference of the generating circle.

(718.) The line A V is called the *axis* of the cycloid, and is equal to the diameter of the generating circle.

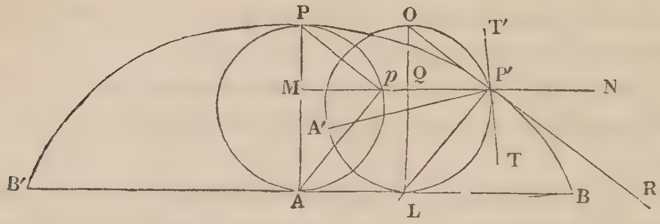
(719.) All lines P P′ parallel to the base and terminated in the cycloid, are bisected at M by the axis; for the branches of the curve at each side of the axis A V are perfectly equal and symmetrical.

(720.) As the generating circle rolls along the base of the cycloid, the describing point P (*fig.* 251.) has two motions; first, a progressive motion in a direction parallel to the base B B′, and secondly, a motion of rotation round the centre of the generating circle.

These motions are equal; for, in the time of one

revolution of the generating circle, the describing point P moves by its progressive motion through the space B B′,

fig. 251.



while by its motion of rotation it moves through a space equal to the circumference of the circle. Let us suppose the circle to roll from the position A in which the describing point P coincides with the vertex of the cycloid, to the position L in which the describing point has moved to P′, and the point which was at A be now at A′. The distance L A will then be equal to the arc L A′ of the circle, since that arc has rolled over L A, and since A B is equal to the semicircle A′ L P′, we have L B equal to the arc of the circle L P′.

The point P′, in virtue of the two equal motions already explained, one in the horizontal direction P′ N, parallel to A B, and the other in the direction of the tangent P′ T to the circle at P′, will have an actual motion in a direction equally inclined to each of these lines. The direction of the curve at P′, or, what is the same, the direction of a tangent to the curve at that point, will therefore be a line bisecting the angle N P′ T. But it is easy to show that such a line will be the continuation of the chord of the arc of the circle between P′ and the highest point O; for if L P′ be drawn, the angle O P′ M will be equal to the angle O L P′, because of the similarity of the right angled triangles O Q P′ and O P′ L, and the angle O P′ T′ will also be equal to the angle O L P′; therefore O P′ T′ will be equal to O P′ Q,

or what is the same, N P′ R will be equal to T P′ R; the line O P′ R therefore bisects the angle T P′ N, and is therefore a tangent to the cycloid at P′.

(721.) Since the arcs A p and L P′ are equal, and also the arcs P p and O P′, the lines A p and L P′ are equal and parallel, and the lines P p and O P′ are likewise equal and parallel.

(722.) The tangent at P′, is therefore parallel to the corresponding chord P p of the generating circle on the axis.

(723.) To draw a tangent therefore to a point P′ on a cycloid, draw a line P′ M from that point perpendicular to the axis A P, and from the point p, where that line meets the generating circle on the axis, draw a chord p P, and through P′ draw a line O P′ R parallel to this chord; this line will be a tangent to the cycloid at P′.

(724.) The arc P p of the generating circle on the axis is equal to the parallel p P′ to the base, intercepted between that arc and the cycloid. For A L has been already proved equal to A′ L; but the latter is equal to O P′, and therefore to P p; but A L is equal to P′ p, being opposite sides of the parallelogram A P′; therefore p P′ is equal to the arc P p.

(725.) It is a property of the cycloid, which may be demonstrated by the aid of the higher analysis, that the cycloidal arc P P′ is equal to twice the chord P p, and this will be the case wherever the parallel P′ p is drawn. Hence the semi-cycloid P B is equal to twice the diameter of the generating circle, and the entire length of the cycloid B P B′ is equal to four times the diameter of the generating circle.

(726.) Hence the length of a cycloid is to the length of its base as four times the diameter of a circle is to its circumference.

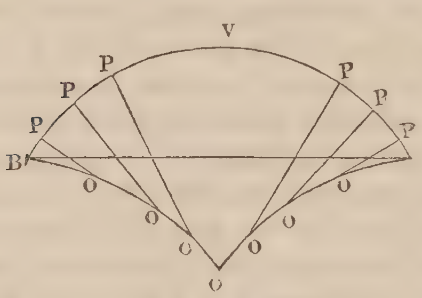
(727.) Since P′ O is a tangent to the cycloid at P′, and the angle O P′ L is a right angle being in a semicircle, the line P′ L is the normal to the cycloid at P′,

and the radius of curvature to the cycloid at P′ is twice P L.

(728.) One of the most remarkable properties of the cycloid is, that it is its own involute; in other words, the involute of a cycloid is an equal and similar cycloid.

Let B B′ (*fig.* 252.) be the base of a cycloid B V B′,

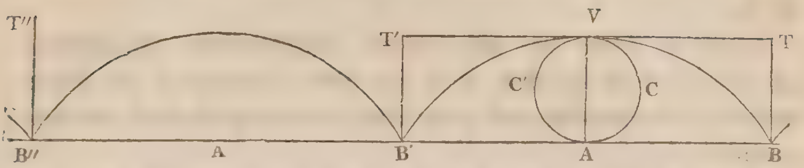
fig. 252.



and let B O′ and B′ O′ be two semi-cycloids, each equal and similar to B V, having their vertices at B and B′. Then B O′ will be the involute of B V, and B′ O′ the involute of B′ V; and in like manner B V will be the evolute of B O′, and B′ V the evolute of B′ O′. If P be any point in the cycloid B V B′, and P O be a tangent to the lower cycloid, then O will be the centre and P O the radius of curvature of the point P, and the lines P O will all be bisected by the base B B′.

(729.) The cycloid, according to the principle by which it is generated, is not supposed to terminate at B and B′, the extremities of the base. For the motion of the generating circle may be continued along the line of the base in either direction beyond these points. If it be so continued, the generating point will describe a succession of cycloidal arcs as represented in *fig.* 253. A *cusp* being formed at the points B, B′, B″ &c., where the describing point touches the base.

fig. 253.



(730. As it has been already stated that the radius of curvature for any point of the cycloid is twice the length of a line drawn from the describing point to the point where the generating circle touches the base, it will be evident that, at the points B, B′, B″, the radius of curvature will vanish, since at these poinis the describing point coincides with the point where the generating circle touches the base.

(731.) By the methods furnished in the higher analysis, it is shown that the area of the generating circle is one third of the area of the cycloid, and consequently it follows, that the space V C A B is two thirds of the space V A B.

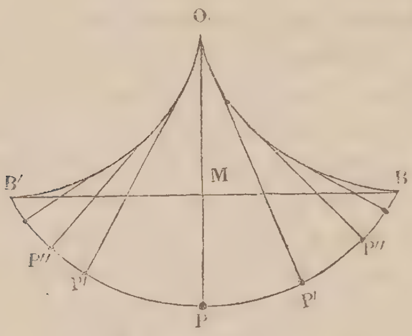
Since the area of the generating circle is equal to a fourth of the rectangle under its diameter and circumference, and the base B B′ is equal to its circumference, it follows that the area of the rectangle B T T′B′ is four times the area of the generating circle, and therefore the area of the cycloid B V B′ is three fourths of that rectangle. Hence the cycloidal arc V B and the lines V T and T B, include a space equal to the semicircle V C A.

(732.) Among the properties which have rendered the cycloid most memorable in the history of science are the following:—

(733.) If a pendulous or heavy body be by any means compelled to move in a curve, as it does in a circular arc when attached to the end of a rod, the other end of which hangs on a fixed point, the times which it takes to vibrate, in arcs of different lengths, are in general unequal. This is the case, for example, in the circle in which the time of the vibration of the pendu-

lum in longer arcs is not the same as in shorter arcs. It was long a question of much curiosity and interest in mathematical physics, to discover the *Isochrone,* or the curve in which all the arcs, whether longer or shorter, would be described by a pendulous body in the same time. This curve was found to be the cycloid. Let (*fig.* 254.) represent two semi-cycloidal faces, and

fig. 254.



O B and O B′ grooves accurately formed to which a flexible string can apply itself. Let B B′ be the line uniting the vertices of the two semi-cycloids, and draw O M perpendicular to B B′, and produce it so that M P shall equal O M. Let O P be a flexible string, to which a weight P is suspended. If the weight P swing alternately to the right and to the left, the string will apply itself alternately to the cycloidal grooves in O B and O B′; and, according to what was explained in (728.) the point P will move in a cycloid, whose base is B B′, whose axis is M P, and whose involutes are O B and O B′. The times of vibration of such a pendulum, whether it vibrates between B and B′ describing the whole cycloidal arc, or from any intermediate points, such as P′ or P″ will be the same.

(734.) It was also a subject of curious physical inquiry, to determine the form of the surface or line down which a body should fall so as to descend from one given point to another given point having a lower position, in the least possible time, or to determine the *brachystochrone*. This curve was likewise found to be the

cycloid, the time of falling from B to P being less than the time of falling through any other line joining the same points B and P.

(735.) It is sometimes stated, though erroneously, that birds, in flying from an elevated to a lower point, proceed in a cycloid; but it should be considered, that the condition of their flight through the atmosphere is extremely different from that of a heavy body moving on a solid resisting surface, or, what is the same, attached to a flexible and inextensible string.

(736.) In defining the cycloid, the pencil by which the curve is traced has been supposed to be on the circumference of the generating circle; it may, however, be *within* or *without* that circle, and corresponding varieties of curves are produced.

If the describing point be outside the generating circle, the curve will be such as is represented in (*fig.* 255.), and is called the *curtate cycloid*. This curve has nodes at A B and D E. The points B and D, where the curve intersects itself, are called *multiple points*.

fig. 255.

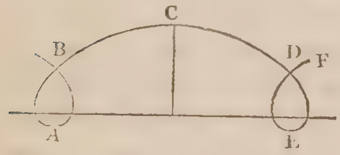
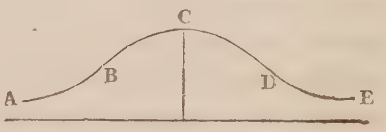


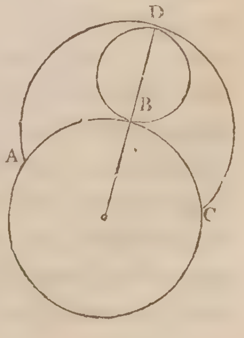
fig. 256.



When the describing point lies within the circle, the curve has a form such as is represented in (*fig.* 256.), and is called the *prolate cycloid*. It has points of inflection at B and D.

(737.) If the generating circle, instead of rolling on a straight line, be supposed to roll upon the circumference of another circle, the curve produced is such as is represented at A D C in (*fig.* 257.), and is called an *epicycloid*.

fig. 257.



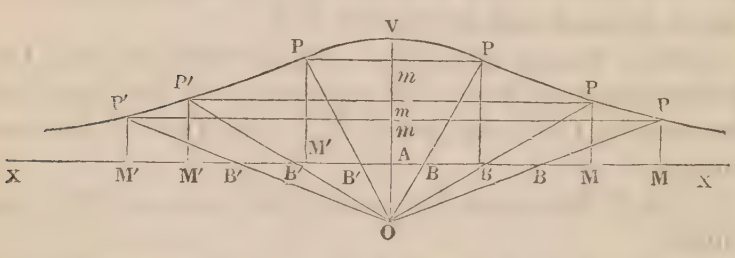
In this case the one circle is supposed to roll outside the other: if it roll within the other a curve is produced within the other called the *hypocycloid.*

THE CONCHOID.

(738.) The conchoid of Nicomedes, so called from the Greek geometer who first conceived this curve, and investigated its form and properties, may be constructed by a series of points in the following manner:—

Let X X′ (*fig.* 258.) be an indefinite straight line,

fig. 258.



and from a point O, called the pole of the conchoid, draw O A perpendicular to it, and produce it to V, so that V A shall have a given length, which we shall call the *parameter* of the curve, the line X X′ being the *directrix.*

From the pole O draw several lines O B, meeting the directrix at B, and produce each of them above the directrix, so that B P, the produced part of each, shall be equal to A V, the parameter of the conchoid. The points P, thus determined, will be points of the curve; and if the lines O B be sufficiently close together, the points P will lie so near each other, that the curve V P P may be drawn through them.

If lines O B′ be in like manner drawn to the directrix on the left of A, points of the curve may be determined, and another branch V P′ P′, may be drawn through them.

(739.) The line O A V divides the curve symmetrically, the branch V P P being in all respects similar to V P′ P′.

For if O B and O B′ be inclined at equal angles to O A, the triangle O A B will be equal to the triangle O A B′; and therefore the line O B will be equal to O B′. But B P and B′ P′ are equal, being both equal to the parameter; therefore O P is equal to O P′; and, since the angle V O P is supposed to be equal to the angle V O P′, the triangle P O m is equal to the triangle P′ O m. Therefore P P′ is bisected at m, and is perpendicular to V O; so that if the curve V P were folded over on V P′, the point P would fall upon P′, and the same would be true of all other corresponding points on each side of the line O V.

(740.) Since, therefore, it appears that all lines P P′ perpendicular to O V, and terminated in the curve, are bisected by O V, the line O V is the axis of the conchoid.

(741.) The point V is called the vertex of the conchoid.

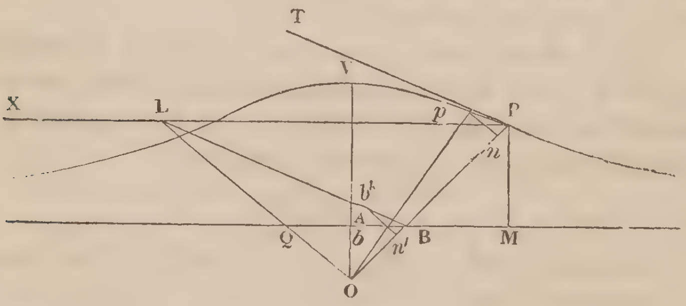
(742.) The directrix X X′ is an asymptote to the two branches of the conchoid.

For, from any point P let a perpendicular P M to X X′ be drawn. The triangle P M B will then be similar to the triangle O A B, and therefore the ratio of P M to P B will be the same as the ratio of O A to O B; but by the definition of the curve, the point B recedes indefinitely from A, and therefore O B is subject to unlimited increase. The ratio therefore of O A to O B is subject to unlimited diminution; but this ratio is the same as that of P M to P B, that is, of the distance of the curve from the directrix to the parameter. Since therefore the ratio of the distance of the curve from the directrix to the parameter is subject to unlimited diminution, the distance of the curve from the directrix will be decreased without limit as the curve recedes from its axis O V, in either direction; the directrix is therefore an asymptote to both branches of the curve.

(743.) To draw a tangent at any point P of the conchoid.

From the pole O (*fig.* 259.) draw O P, and take a point *p* on the curve so near P that the arc P *p* may be regarded as a straight line, and draw O *p* crossing the directrix at *b*. From *p* and *b* draw *p n* and *b n′* perpendicular to O B P. Since B P is equal to *b p*, the difference between O P and O *p* is equal to the difference between O B and O *b*. But since *pn* and *b n′* may be considered as small circular arcs, of which O is the centre, these differences will be P *n* and B *n′*: we shall have therefore P *n* equal to B *n′*. Again as the triangles *b* O *n′* and *p* O *n* are similar, *b n′* is to *p n* as O *n′* is to O *n*, or as O B is to O P, because B *n′* and P *n* being indefinitely small, O B and O P may be taken instead of O *n′* and O *n*. Draw P X parallel to A M, O L perpendicular to OP, and join B L, meeting the production *n′ b* in *b′*; then *n′ b′* will be parallel O L, and therefore *b n′* is to *b′ n′* as O Q is to O L, or as O B is to O P, or as *b n′* is to *p n*, by what has been proved before; we hence infer that *b′ n′* is equal to *p n*, and since B *n′* is equal to P *n*, the angle *b′* B *n′* will be equal to the angle *p* P *n*, and the line B *b′* or B L will be parallel to the curve, or to its tangent at P.

fig. 259.

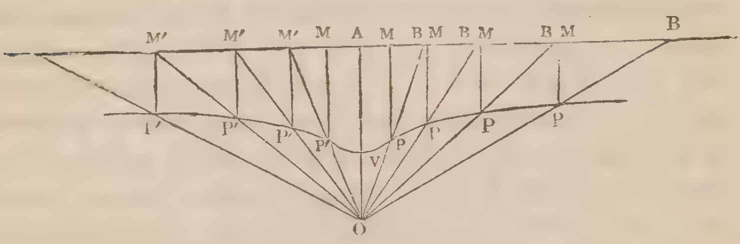


To draw a tangent at P, therefore, draw P X parallel to the directrix, join O P, and through O draw O L perpendicular to O P. From L draw L B to the point where P O crosses the directrix, and from P draw P T parallel to B L; the line P T will then be the required tangent.

(744.) If, instead of producing the line O B above the directrix till the produced part is equal to the parameter, a part be taken upon O B, from B towards O, equal to the parameter; the points thus determined will lie in a curve, having properties, similar to those of the conchoid already described.

(745.) Let O A (*fig.* 260.) be first supposed to be greater than the parameter, and take A V upon it equal

fig. 260.

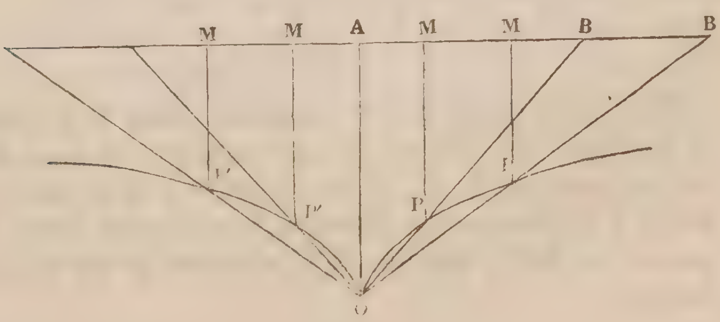


to the parameter. From O, as in the former case, draw any number of lines O B from the pole to the directrix, and take upon them severally distances B P equal to the parameter; a curve being drawn through the point P, thus determined, is called the inferior conchoid.

(746.) It may be proved that O A is an axis of the curve, and that the branches V P P and V P′ P′ on each side of it are similar, and that the directrix is an asymptote, by the same reasoning as has been used in the case of the superior conchoid.

(747.) Both these curves are concave towards the directrix at the vertex, and on each side of it, but afterwards become convex towards it, by passing through a point of inflexion or contrary flexure.

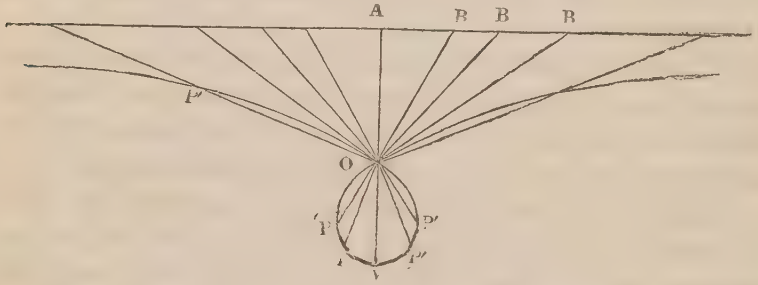
fig. 261.



(748.) If the parameter of the inferior conchoid be equal to the distance of the pole from the directrix, the branches of the curve will then be everywhere convex towards the directrix. This case of the inferior conchoid is represented in *fig.* 261. The form of the curve will be easily traced from the conditions under which the point P are determined. The two branches of the curve meet at the pole O, where they form a cusp.

(749.) If the parameter of the inferior conchoid be greater than the distance of the pole from the directrix,

fig. 262.



the curve will still pass through its pole, but will form a node. This case of the inferior conchoid is represented in *fig.* 262.

THE CATENARY.

(750.) The curve in which a perfectly flexible cord or chain hangs when it is suspended by two points that are not in the same vertical line, is called a catenary. Let A and B (*fig.* 263.) be the two points of suspension, and let A V P B be the catenary formed by a cord suspended at these points.

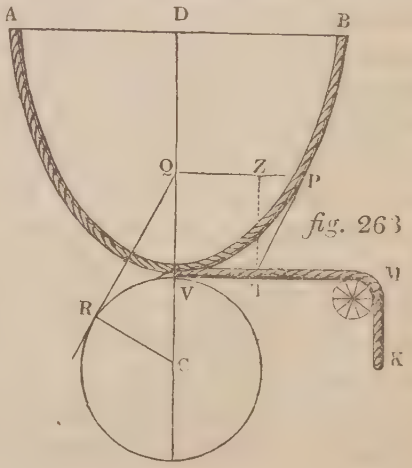


fig. 263

(751.) The weight of the cord produces at each point of it a certain tension, which is balanced by the strength of the cord. Let V be the lowest point of the curve, and supposing the cord V P B to be cut off from A V, and a similar cord being attached to it at V and carried in a horizontal direction over a pulley M, let M K be such a length of the cord that its weight shall be just sufficient to keep the cord A V in the position which it had when connected with the point B. It is evident that the weight of M K, will be the tension of the catenary at the lowest point V.

(752.) The length M K is called the parameter of the catenary.

(753.) If the points A and B be in the same horizontal line, the line D V, drawn in the vertical direction from the middle point of A B, will be the axis of the catenary, and will divide the curve symmetrically.

(754.) Produce D V downwards, so that V C shall be equal to the parameter of the curve, and with C as centre, and C V as radius, let a circle be described. From any point P in the catenary, draw P Q perpendicular to D V, and from Q draw Q R a tangent to the circle. A line P T drawn from P parallel to Q R, will then be a tangent to the catenary at P.

(755.) The catenary being defined by mechanical qualities, its properties must necessarily be derived from mechanical laws. It follows, from the nature of the centre of gravity as demonstrated in mechanics, that the tensions at V and P are in equilibrium with the weight of the cord between P and V acting vertically at its centre of gravity. If T Z be then parallel to V D, a triangle whose sides are parallel to T P, T V, and T Z will represent these three forces, viz. the tension at P, the tension at V, and the weight of P V. Such a triangle is Z P T. If, therefore, Z P represent the tension at V, and Z T the weight of P V, P T will represent the tension at P.

(756.) Since P T is greater than P Z, the strain or tension at P will be greater than the strain at V, and

the same being true for every point from V to B, it follows that the strain on a catenary is least at its lowest point.

(757.) Since Q R is parallel to P T, and Q C to T Z, the angle R Q C will be equal to the angle Z T P, and the triangle R Q C will therefore be similar to the triangle Z T P. Since the sides of the latter represent the tensions at P and V, and the weight of the cord P V, these forces will be likewise represented by the sides of the triangle R Q C; but since R C is the length of the cord whose weight is equal to the tension at V, R Q must be equal to the cord V P, and C Q will be the length of the cord whose weight would represent the tension at P.

(758.) If a tangent be drawn to the circle from D, this tangent will be equal to the cord B P V, and will be parallel to the tangent to the catenary at B. Such will be the direction of the strain upon the point B.

(759.) When P coincides with B, Q will coincide with D, and Q C will become equal to D C. Since in general Q C is the length of the cord whose weight expresses the tension at P, D C will be the length of the cord whose weight is equal to the strain upon the point B.

TABLE

OF

SQUARES, CUBES, SQUARE ROOTS, AND CUBE ROOTS,

OF ALL NUMBERS

FROM 1 TO 1000.

Num.	Square.	Cube.	Squ. Root.	Cube Root.
1	1	1	1·0000000	1·000000
2	4	8	1·4142136	1·259921
3	9	27	1·7320508	1·442250
4	16	64	2·0000000	1·587401
5	25	125	2·2360680	1·709976
6	36	216	2·4494897	1·817121
7	49	343	2·6457513	1·912931
8	64	512	2·8284271	2·000000
9	81	729	3·0000000	2·080084
10	1 00	1 000	3·1622777	2·154435
11	1 21	1 331	3·3166248	2·223980
12	1 44	1 728	3·4641016	2·289429
13	1 69	2 197	3·6055513	2·351335
14	1 96	2 744	3·7416574	2·410142
15	2 25	3 375	3·8729833	2·466212
16	2 56	4 096	4·0000000	2·519842
17	2 89	4 913	4·1231056	2·571282
18	3 24	5 832	4·2426407	2·620741
19	3 61	6 859	4·3588989	2·668402
20	4 00	8 000	4·4721360	2·714418
21	4 41	9 261	4·5825757	2·758924
22	4 84	10 648	4·6904158	2·802039
23	5 29	12 167	4·7958315	2·843867
24	5 76	13 824	4·8989795	2·884499
25	6 25	15 625	5·0000000	2·924018
26	6 76	17 576	5·0990195	2·962496
27	7 29	19 683	5·1961524	3·000000
28	7 84	21 952	5·2915026	3·036589
29	8 41	24 389	5·3851648	3·072317
30	9 00	27 000	5·4772256	3·107232
31	9 61	29 791	5·5677644	3·141381
32	10 24	32 768	5·6568542	3·174802
33	10 89	35 937	5·7445626	3·207534
34	11 56	39 304	5·8309519	3·239612
35	12 25	42 875	5·9160798	3·271066
36	12 96	46 656	6·0000000	3·301927
37	13 69	50 653	6·0827625	3·332222
38	14 44	54 872	6·1644140	3·361975
39	15 21	59 319	6·2449980	3·391211
40	16 00	64 000	6·3245553	3·419952
41	16 81	68 921	6·4031242	3·448217
42	17 64	74 088	6·4807407	3·476027
43	18 49	79 507	6·5574385	3·503398
44	19 36	85 184	6·6332496	3·530348
45	20 25	91 125	6·7082039	4·556893

Num.	Square.	Cube.	Squ. Root.	Cube Root.
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47	22 09	103 823	6·8556546	3·608826
48	23 04	110 592	6·9282032	3·634241
49	24 01	117 649	7·0000000	3·659306
50	25 00	125 000	7·0710678	3·684031
51	26 01	132 651	7·1414284	3·708430
52	27 04	140 608	7·2111026	3·732511
53	28 09	148 877	7·2801099	3·756286
54	29 16	157 464	7·3484692	3·779763
55	30 25	166 375	7·4161985	3·802953
56	31 36	175 616	7·4833148	3·825862
57	32 49	185 193	7·5498344	3·848501
58	33 64	195 112	7·6157731	3·870877
59	34 81	205 379	7·6811457	3·892996
60	36 00	216 000	7·7459667	3·914868
61	37 21	226 981	7·8102497	3·936497
62	38 44	238 328	7·8740079	3·957892
63	39 69	250 047	7·9372539	3·979057
64	40 96	262 144	8·0000000	4·000000
65	42 25	274 625	8·0622577	4·020726
66	43 56	287 496	8·1240384	4·041240
67	44 89	300 763	8·1853528	4·061548
68	46 24	314 432	8·2462113	4·081655
69	47 61	328 509	8·3066239	4·101566
70	49 00	343 000	8·3666003	4·121285
71	50 41	357 911	8·4261498	4·140818
72	51 84	373 248	8·4852814	4·160168
73	53 29	389 017	8·5440037	4·179339
74	54 76	405 224	8·6023253	4·198336
75	56 25	421 875	8·6602540	4·217163
76	57 76	438 976	8·7177979	4·235824
77	59 29	456 533	8·7749644	4·254321
78	60 84	474 552	8·8317609	4·272659
79	62 41	493 039	8·8881944	4·290840
80	64 00	512 000	8·9442719	4·308870
81	65 61	531 441	9·0000000	4·326749
82	67 24	551 368	9·0553851	4·344481
83	68 89	571 787	9·1104336	4·362071
84	70 56	592 704	9·1651514	4·379519
85	72 25	614 125	9·2195445	4·396830
86	73 96	636 056	9·2736185	4·414005
87	75 69	658 503	9·3273791	4·431048
88	77 44	681 472	9·3808315	4·447960
89	79 21	704 969	9·4339811	4·464745
90	81 00	729 000	9·4868330	4·481405

Num.	Square.	Cube.	Squ. Root.	Cube Root.
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93	86 49	804 357	9·6436508	4·530655
94	88 36	830 584	9·6953597	4·546836
95	90 25	857 375	9·7467943	4·562903
96	92 16	884 736	9·7979590	4·578857
97	94 09	912 673	9·8488578	4·594701
98	96 04	941 192	9·8994949	4·610436
99	98 01	970 299	9·9498744	4·626065
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101	1 02 01	1 030 301	10·0498756	4·657010
102	1 04 04	1 061 208	10·0995049	4·672329
103	1 06 09	1 092 727	10·1488916	4·687548
104	1 08 16	1 124 864	10·1980390	4·702669
105	1 10 25	1 157 625	10·2469508	4·717694
106	1 12 36	1 191 016	10·2956301	4·732624
107	1 14 49	1 225 043	10·3440804	4·747459
108	1 16 64	1 259 712	10·3923048	4·762203
109	1 18 81	1 295 029	10·4403065	4·776856
110	1 21 00	1 331 000	10·4880885	4·791420
111	1 23 21	1 367 631	10·5356538	4·805896
112	1 25 44	1 404 928	10·5830052	4·820284
113	1 27 69	1 442 897	10·6301458	4·834588
114	1 29 96	1 481 544	10·6770783	4·848808
115	1 32 25	1 520 875	10·7238053	4·862944
116	1 34 56	1 560 896	10·7703296	4·876999
117	1 36 89	1 601 613	10·8166538	4·890973
118	1 39 24	1 643 032	10·8627805	4·904868
119	1 41 61	1 685 159	10·9087121	4·918685
120	1 44 00	1 728 000	10·9544512	4·932424
121	1 46 41	1 771 561	11·0000000	4·946087
122	1 48 84	1 815 848	11·0453610	4·959676
123	1 51 29	1 860 867	11·0905365	4·973190
124	1 53 76	1 906 624	11·1355287	4·986631
125	1 56 25	1 953 125	11·1803399	5·000000
126	1 58 76	2 000 376	11·2249722	5·013298
127	1 61 29	2 048 383	11·2694277	5·026526
128	1 63 84	2 097 152	11·3137085	5·039684
129	1 66 41	2 146 689	11·3578167	5·052774
130	1 69 00	2 197 000	11·4017543	5·065797
131	1 71 61	2 248 091	11·4455231	5·078753
132	1 74 24	2 299 968	11·4891253	5·091643
133	1 76 89	2 352 637	11·5325626	5·104469
134	1 79 56	2 406 104	11·5758369	5·117230
135	1 82 25	2 460 375	11·6189500	5·129928

Num.	Square.	Cube.	Squ. Root.	Cube Root.
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137	18769	2571353	11·7046999	5·155137
138	19044	2628072	11·7473401	5·167649
139	19321	2685619	11·7898261	5·180101
140	19600	2744000	11·8321596	5·192494
141	19881	2803221	11·8743421	5·204828
142	20164	2863288	11·9163753	5·217103
143	20449	2924207	11·9582607	5·229321
144	20736	2985984	12·0000000	5·241483
145	21025	3048625	12·0415946	5·253588
146	21316	3112136	12·0830460	5·265637
147	21609	3176523	12·1243557	5·277632
148	21904	3241792	12·1655251	5·289572
149	22201	3307949	12·2065556	5·301459
150	22500	3375000	12·2474487	5·313293
151	22801	3442951	12·2882057	5·325074
152	23104	3511808	12·3288280	5·336803
153	23409	3581577	12·3693169	5·348481
154	23716	3652264	12·4096736	5·360108
155	24025	3723875	12·4498996	5·371685
156	24336	3796416	12·4899960	5·383213
157	24649	3869893	12·5299641	5·394691
158	24964	3944312	12·5698051	5·406120
159	25281	4019679	12·6095202	5·417501
160	25600	4096000	12·6491106	5·428835
161	25921	4173281	12·6885775	5·440122
162	26244	4251528	12·7279221	5·451362
163	26569	4330747	12·7671453	5·462556
164	26896	4410944	12·8062485	5·473704
165	27225	4492125	12·8452326	5·484807
166	27556	4574296	12·8840987	5·495865
167	27889	4657463	12·9228480	5·506878
168	28224	4741632	12·9614814	5·517848
169	28561	4826809	13·0000000	5·528775
170	28900	4913000	13·0384048	5·539658
171	29241	5000211	13·0766968	5·550499
172	29584	5088448	13·1148770	5·561298
173	29929	5177717	13·1529464	5·572055
174	30276	5268024	13·1909060	5·582770
175	30625	5359375	13·2287566	5·593445
176	30976	5451776	13·2664992	5·604079
177	31329	5545233	13·3041347	5·614672
178	31684	5639752	13·3416641	5·625226
179	32041	5735339	13·3790882	5·635741
180	32400	5832000	13·4164079	5·646216

Num.	Square.	Cube.	Squ. Root.	Cube Root.
181	3 27 61	5 929 741	13·4536240	5·656653
182	3 31 24	6 028 568	13·4907376	5·667051
183	3 34 89	6 128 487	13·5277493	5·677411
184	3 38 56	6 229 504	13·5646600	5·687734
185	3 42 25	6 331 625	13·6014705	5·698019
186	3 45 96	6 434 856	13·6381817	5·708267
187	3 49 69	6 539 203	13·6747943	5·718479
188	3 53 44	6 644 672	13·7113092	5·728654
189	3 57 21	6 751 269	13·7477271	5·738794
190	3 61 00	6 859 000	13·7840488	5·748897
191	3 64 81	6 967 871	13·8202750	5·758965
192	3 68 64	7 077 888	13·8564065	5·768998
193	3 72 49	7 189 057	13·8924440	5·778997
194	3 76 36	7 301 384	13·9283883	5·788960
195	3 80 25	7 414 875	13·9642400	5·798890
196	3 84 16	7 529 536	14·0000000	5·808786
197	3 88 09	7 645 373	14·0356688	5·818648
198	3 92 04	7 762 392	14·0712473	5·828477
199	3 96 01	7 880 599	14·1067360	5·838272
200	4 00 00	8 000 000	14·1421356	5·848035
201	4 04 01	8 120 601	14·1774469	5·857766
202	4 08 04	8 242 408	14·2126704	5·867464
203	4 12 09	8 365 427	14·2478068	5·877131
204	4 16 16	8 489 664	14·2828569	5·886765
205	4 20 25	8 615 125	14·3178211	5·896368
206	4 24 36	8 741 816	14·3527001	5·905941
207	4 28 49	8 869 743	14·3874946	5·915482
208	4 32 64	8 998 912	14·4222051	5·924992
209	4 36 81	9 129 329	14·4568323	5·934472
210	4 41 00	9 261 000	14·4913767	5·943922
211	4 45 21	9 393 931	14·5258390	5·953342
212	4 49 44	9 528 128	14·5602198	5·962732
213	4 53 69	9 663 597	14·5945195	5·972093
214	4 57 96	9 800 344	14·6287388	5·981424
215	4 62 25	9 938 375	14·6628783	5·990726
216	4 66 56	10 077 696	14·6969385	6·000000
217	4 70 89	10 218 313	14·7309199	6·009245
218	4 75 24	10 360 232	14·7648231	6·018462
219	4 79 61	10 503 459	14·7986486	6·027650
220	4 84 00	10 648 000	14·8323970	6·036811
221	4 88 41	10 793 861	14·8660687	6·045943
222	4 92 84	10 941 048	14·8996644	6·055049
223	4 97 29	11 089 567	14·9331845	6·064127
224	5 01 76	11 239 424	14·9666295	6·073178
225	5 06 25	11 390 625	15·0000000	6·082202

Num.	Square.	Cube.	Squ. Root.	Cube Root.
226	51076	11543176	15·0332964	6·091199
227	51529	11697083	15·0665192	6·100170
228	51984	11852352	15·0996689	6·109115
229	52441	12008989	15·1327460	6·118033
230	52900	12167000	15·1657509	6·126926
231	53361	12326391	15·1986842	6·135792
232	53824	12487168	15·2315462	6·144634
233	54289	12649337	15·2643375	6·153449
234	54756	12812904	15·2970585	6·162240
235	55225	12977875	15·3297097	6·171006
236	55696	13144256	15·3622915	6·179746
237	56169	13312053	15·3948043	6·188463
238	56644	13481272	15·4272486	6·197154
239	57121	13651919	15·4596248	6·205822
240	57600	13824000	15·4919334	6·214465
241	58081	13997521	15·5241747	6·223084
242	58564	14172488	15·5563492	6·231680
243	59049	14348907	15·5884573	6·240251
244	59536	14526784	15·6204994	6·248800
245	60025	14706125	15·6524758	6·257325
246	60516	14886936	15·6843871	6·265827
247	61009	15069223	15·7162336	6·274305
248	61504	15252992	15·7480157	6·282761
249	62001	15438249	15·7797338	6·291195
250	62500	15625000	15·8113883	6·299605
251	63001	15813251	15·8429795	6·307994
252	63504	16003008	15·8745079	6·316360
253	64009	16194277	15·9059737	6·324704
254	64516	16387064	15·9373775	6·333026
255	65025	16581375	15·9687194	6·341326
256	65536	16777216	16·0000000	6·349604
257	66049	16974593	16·0312195	6·357861
258	66564	17173512	16·0623784	6·366097
259	67081	17373979	16·0934769	6·374311
260	67600	17576000	16·1245155	6·382504
261	68121	17779581	16·1554944	6·390676
262	68644	17984728	16·1864141	6·398828
263	69169	18191447	16·2172747	6·406958
264	69696	18399744	16·2480768	6·415069
265	70225	18609625	16·2788206	6·423158
266	70756	18821096	16·3095064	6·431228
267	71289	19034163	16·3401346	6·439277
268	71824	19248832	16·3707055	6·447306
269	72361	19465109	16·4012195	6·455315
270	72900	19683000	16·4316767	6·463304

Num.	Square.	Cube.	Squ. Root.	Cube Root.
271	7 34 41	19 902 511	16·4620776	6·471274
272	7 39 84	20 123 648	16·4924225	6·479224
273	7 45 29	20 346 417	16·5227116	6·487154
274	7 50 76	20 570 824	16·5529454	6·495065
275	7 56 25	20 796 875	16·5831240	6·502957
276	7 61 76	21 024 576	16·6132477	6·510830
277	7 67 29	21 253 933	16·6433170	6·518684
278	7 72 84	21 484 952	16·6733320	6·526519
279	7 78 41	21 717 639	16·7032931	6·534335
280	7 84 00	21 952 000	16·7332005	6·542133
281	7 89 61	22 188 041	16·7630546	6·549912
282	7 95 24	22 425 768	16·7928556	6·557672
283	8 00 89	22 665 187	16·8226038	6·565414
284	8 06 56	22 906 304	16·8522995	6·573139
285	8 12 25	23 149 125	16·8819430	6·580844
286	8 17 96	23 393 656	16·9115345	6·588532
287	8 23 69	23 639 903	16·9410743	6·596202
288	8 29 44	23 887 872	16·9705627	6·603854
289	8 35 21	24 137 569	17·0000000	6·611489
290	8 41 00	24 389 000	17·0293864	6·619106
291	8 46 81	24 642 171	17·0587221	6·626705
292	8 52 64	24 897 088	17·0880075	6·634287
293	8 58 49	25 153 757	17·1172428	6·641852
294	8 64 36	25 412 184	17·1464282	6·649400
295	8 70 25	25 672 375	17·1755640	6·656930
296	8 76 16	25 934 336	17·2046505	6·664444
297	8 82 09	26 198 073	17·2336879	6·671940
298	8 88 04	26 463 592	17·2626765	6·679420
299	8 94 01	26 730 899	17·2916165	6·686883
300	9 00 00	27 000 000	17·3205081	6·694329
301	9 06 01	27 270 901	17·3493516	6·701759
302	9 12 04	27 543 608	17·3781472	6·709173
303	9 18 09	27 818 127	17·4068952	6·716570
304	9 24 16	28 094 464	17·4355958	6·723951
305	9 30 25	28 372 625	17·4642492	6·731316
306	9 36 36	28 652 616	17·4928557	6·738664
307	9 42 49	28 934 443	17·5214155	6·745997
308	9 48 64	29 218 112	17·5499288	6·753313
309	9 54 81	29 503 629	17·5783958	6·760614
310	9 61 00	29 791 000	17·6068169	6·767899
311	9 67 21	30 080 231	17·6351921	6·775169
312	9 73 44	30 371 328	17·6635217	6·782423
313	9 79 69	30 664 297	17·6918060	6·789661
314	9 85 96	30 959 144	17·7200451	6·796884
315	9 92 25	31 255 875	17·7482393	6·804092

Num.	Square.	Cube.	Squ. Root.	Cube Root.
316	9 98 56	31 554 496	17·7763888	6·811285
317	10 04 89	31 855 013	17·8044938	6·818462
318	10 11 24	32 157 432	17·8325545	6·825624
319	10 17 61	32 461 759	17·8605711	6·832771
320	10 24 00	32 768 000	17·8885438	6·839904
321	10 30 41	33 076 161	17·9164729	6·847021
322	10 36 84	33 386 248	17·9443584	6·854124
323	10 43 29	33 698 267	17·9722008	6·861212
324	10 49 76	34 012 224	18·0000000	6·868285
325	10 56 25	34 328 125	18·0277564	6·875344
326	10 62 76	34 645 976	18·0554701	6·882389
327	10 69 29	34 965 783	18·0831413	6·889419
328	10 75 84	35 287 552	18·1107703	6·896435
329	10 82 41	35 611 289	18·1383571	6·903436
330	10 89 00	35 937 000	18·1659021	6·910423
331	10 95 61	36 264 691	18·1934054	6·917396
332	11 02 24	36 594 368	18·2208672	6·924356
333	11 08 89	36 926 037	18·2482876	6·931301
334	11 15 56	37 259 704	18·2756669	6·938232
335	11 22 25	37 595 375	18·3030052	6·945150
336	11 28 96	37 933 056	18·3303028	6·952053
337	11 35 69	38 272 753	18·3575598	6·958943
338	11 42 44	38 614 472	18·3847763	6·965820
339	11 49 21	38 958 219	18·4119526	6·972683
340	11 56 00	39 304 000	18·4390889	6·979532
341	11 62 81	39 651 821	18·4661853	6·986368
342	11 69 64	40 001 688	18·4932420	6·993191
343	11 76 49	40 353 607	18·5202592	7·000000
344	11 83 36	40 707 584	18·5472370	7·006796
345	11 90 25	41 063 625	18·5741756	7·013579
346	11 97 16	41 421 736	18·6010752	7·020349
347	12 04 09	41 781 923	18·6279360	7·027106
348	12 11 04	42 144 192	18·6547581	7·033850
349	12 18 01	42 508 549	18·6815417	7·040581
350	12 25 00	42 875 000	18·7082869	7·047299
351	12 32 01	43 243 551	18·7349940	7·054004
352	12 39 04	43 614 208	18·7616630	7·060697
353	12 46 09	43 986 977	18·7882942	7·067377
354	12 53 16	44 361 864	18·8148877	7·074044
355	12 60 25	44 738 875	18·8414437	7·080699
356	12 67 36	45 118 016	18·8679623	7·087341
357	12 74 49	45 499 293	18·8944436	7·093971
358	12 81 64	45 882 712	18·9208879	7·100588
359	12 88 81	46 268 279	18·9472953	7·107194
360	12 96 00	46 656 000	18·9736660	7·113787

Num.	Square.	Cube.	Squ. Root.	Cube Root.
361	13 03 21	47 045 881	19·0000000	7·120367
362	13 10 44	47 437 928	19·0262976	7·126936
363	13 17 69	47 832 147	19·0525589	7·133492
364	13 24 96	48 228 544	19·0787840	7·140037
365	13 32 25	48 627 125	19·1049732	7·146569
366	13 39 56	49 027 896	19·1311265	7·153090
367	13 46 89	49 430 863	19·1572441	7·159599
368	13 54 24	49 836 032	19·1833261	7·166096
369	13 61 61	50 243 409	19·2093727	7·172581
370	13 69 00	50 653 000	19·2353841	7·179054
371	13 76 41	51 064 811	19·2613603	7·185516
372	13 83 84	51 478 848	19·2873015	7·191966
373	13 91 29	51 895 117	19·3132079	7·198405
374	13 98 76	52 313 624	19·3390796	7·204832
375	14 06 25	52 734 375	19·3649167	7·211248
376	14 13 76	53 157 376	19·3907194	7·217652
377	14 21 29	53 582 633	19·4164878	7·224045
378	14 28 84	54 010 152	19·4422221	7·230427
379	14 36 41	54 439 939	19·4679223	7·236797
380	14 44 00	54 872 000	19·4935887	7·243156
381	14 51 61	55 306 341	19·5192213	7·249504
382	14 59 24	55 742 968	19·5448203	7·255841
383	14 66 89	56 181 887	19·5703858	7·262167
384	14 74 56	56 623 104	19·5959179	7·268482
385	14 82 25	57 066 625	19·6214169	7·274786
386	14 89 96	57 512 456	19·6468827	7·281079
387	14 97 69	57 960 603	19·6723156	7·287362
388	15 05 44	58 411 072	19·6977156	7·293633
389	15 13 21	58 863 869	19·7230829	7·299894
390	15 21 00	59 319 000	19·7484177	7·306144
391	15 28 81	59 776 471	19·7737199	7·312383
392	15 36 64	60 236 288	19·7989899	7·318611
393	15 44 49	60 698 457	19·8242276	7·324829
394	15 52 36	61 162 984	19·8494332	7·331037
395	15 60 25	61 629 875	19·8746069	7·337234
396	15 68 16	62 099 136	19·8997487	7·343420
397	15 76 09	62 570 773	19·9248588	7·349597
398	15 84 04	63 044 792	19·9499373	7·355762
399	15 92 01	63 521 199	19·9749844	7·361918
400	16 00 00	64 000 000	20·0000000	7·368063
401	16 08 01	64 481 201	20·0249844	7·374198
402	16 16 04	64 964 808	20·0499377	7·380323
403	16 24 09	65 450 827	20·0748599	7·386437
404	16 32 16	65 939 264	20·0997512	7·392542
405	16 40 25	66 430 125	20·1246118	7·398636

Num.	Square.	Cube.	Squ. Root.	Cube Root.
406	16 48 36	66 923 416	20·1494417	7·404721
407	16 56 49	67 419 143	20·1742410	7·410795
408	16 64 64	67 917 312	20·1990099	7·416859
409	16 72 81	68 417 929	20·2237484	7·422914
410	16 81 00	68 921 000	20·2484567	7·428959
411	16 89 21	69 426 531	20·2731349	7·434994
412	16 97 44	69 934 528	20·2977831	7·441019
413	17 05 69	70 444 997	20·3224014	7·447034
414	17 13 96	70 957 944	20·3469899	7·453040
415	17 22 25	71 473 375	20·3715488	7·459036
416	17 30 56	71 991 296	20·3960781	7·465022
417	17 38 89	72 511 713	20·4205779	7·470999
418	17 47 24	73 034 632	20·4450483	7·476966
419	17 55 61	73 560 059	20·4694895	7·482924
420	17 64 00	74 088 000	20·4939015	7·488872
421	17 72 41	74 618 461	20·5182845	7·494811
422	17 80 84	75 151 448	20·5426386	7·500741
423	17 89 29	75 686 967	20·5669638	7·506661
424	17 97 76	76 225 024	20·5912603	7·512571
425	18 06 25	76 765 625	20·6155281	7·518473
426	18 14 76	77 308 776	20·6397674	7·524365
427	18 23 29	77 854 483	20·6639783	7·530248
428	18 31 84	78 402 752	20·6881609	7·536122
429	18 40 41	78 953 589	20·7123152	7·541987
430	18 49 00	79 507 000	20·7364414	7·547842
431	18 57 61	80 062 991	20·7605395	7·553689
432	18 66 24	80 621 568	20·7846097	7·559526
433	18 74 89	81 182 737	20·8086520	7·565355
434	18 83 56	81 746 504	20·8326667	7·571174
435	18 92 25	82 312 875	20·8566536	7·576985
436	19 00 96	82 881 856	20·8806130	7·582786
437	19 09 69	83 453 453	20·9045450	7·588579
438	19 18 44	84 027 672	20·9284495	7·594363
439	19 27 21	84 604 519	20·9523268	7·600138
440	19 36 00	85 184 000	20·9761770	7·605905
441	19 44 81	85 766 121	21·0000000	7·611663
442	19 53 64	86 350 888	21·0237960	7·617412
443	19 62 49	86 938 307	21·0475652	7·623152
444	19 71 36	87 528 384	21·0713075	7·628884
445	19 80 25	88 121 125	21·0950231	7·634607
446	19 89 16	88 716 536	21·1187121	7·640321
447	19 98 09	89 314 623	21·1423745	7·646027
448	20 07 04	89 915 392	21·1660105	7·651725
449	20 16 01	90 518 849	21·1896201	7·657414
450	20 25 00	91 125 000	21·2132034	7·663094

Num.	Square.	Cube.	Squ. Root.	Cube Root.
451	20 34 01	91 733 851	21·2367606	7·668766
452	20 43 04	92 345 408	21·2602916	7·674430
453	20 52 09	92 959 677	21·2837967	7·680086
454	20 61 16	93 576 664	21·3072758	7·685733
455	20 70 25	94 196 375	21·3307290	7·691372
456	20 79 36	94 818 816	21·3541565	7·697002
457	20 88 49	95 443 993	21·3775583	7·702625
458	20 97 64	96 071 912	21·4009346	7·708239
459	21 06 81	96 702 579	21·4242853	7·713845
460	21 16 00	97 336 000	21·4476106	7·719443
461	21 25 21	97 972 181	21·4709106	7·725032
462	21 34 44	98 611 128	21·4941853	7·730614
463	21 43 69	99 252 847	21·5174348	7·736188
464	21 52 96	99 897 344	21·5406592	7·741753
465	21 62 25	100 544 625	21·5638587	7·747311
466	21 71 56	101 194 696	21·5870331	7·752861
467	21 80 89	101 847 563	21·6101828	7·758402
468	21 90 24	102 503 232	21·6333077	7·763936
469	21 99 61	103 161 709	21·6564078	7·769462
470	22 09 00	103 823 000	21·6794834	7·774980
471	22 18 41	104 487 111	21·7025344	7·780490
472	22 27 84	105 154 048	21·7255610	7·785993
473	22 37 29	105 823 817	21·7485632	7·791487
474	22 46 76	106 496 424	21·7715411	7·796974
475	22 56 25	107 171 875	21·7944947	7·802454
476	22 65 76	107 850 176	21·8174242	7·807925
477	22 75 29	108 531 333	21·8403297	7·813389
478	22 84 84	109 215 352	21·8632111	7·818846
479	22 94 41	109 902 239	21·8860686	7·824294
480	23 04 00	110 592 000	21·9089023	7·829735
481	23 13 61	111 284 641	21·9317122	7·835169
482	23 23 24	111 980 168	21·9544984	7·840595
483	23 32 89	112 678 587	21·9772610	7·846013
484	23 42 56	113 379 904	22·0000000	7·851424
485	23 52 25	114 084 125	22·0227155	7·856828
486	23 61 96	114 791 256	22·0454077	7·862224
487	23 71 69	115 501 303	22·0680765	7·867613
488	23 81 44	116 214 272	22·0907220	7·872994
489	23 91 21	116 930 169	22·1133444	7·878368
490	24 01 00	117 649 000	22·1359436	7·883735
491	24 10 81	118 370 771	22·1585198	7·889095
492	24 20 64	119 095 488	22·1810730	7·894447
493	24 30 49	119 823 157	22·2036033	7·899792
494	24 40 36	120 553 784	22·2261108	7·905129
495	24 50 25	121 287 375	22·2485955	7·910460

Num.	Square.	Cube.	Squ. Root.	Cube Root.
496	24 60 16	122 023 936	22·2710575	7·915783
497	24 70 09	122 763 473	22·2934968	7·921099
498	24 80 04	123 505 992	22·3159136	7·926408
499	24 90 01	124 251 499	22·3383079	7·931710
500	25 00 00	125 000 000	22·3606798	7·937005
501	25 10 01	125 751 501	22·3830293	7·942293
502	25 20 04	126 506 008	22·4053565	7·947574
503	25 30 09	127 263 527	22·4276615	7·952848
504	25 40 16	128 024 064	22·4499443	7·958114
505	25 50 25	128 787 625	22·4722051	7·963374
506	25 60 36	129 554 216	22·4944438	7·968627
507	25 70 49	130 323 843	22·5166605	7·973873
508	25 80 64	131 096 512	22·5388553	7·979112
509	25 90 81	131 872 229	22·5610283	7·984344
510	26 01 00	132 651 000	22·5831796	7·989570
511	26 11 21	133 432 831	22·6053091	7·994788
512	26 21 44	134 217 728	22·6274170	8·000000
513	26 31 69	135 005 697	22·6495033	8·005205
514	26 41 96	135 796 744	22·6715681	8·010403
515	26 52 25	136 590 875	22·6936114	8·015595
516	26 62 56	137 388 096	22·7156334	8·020779
517	26 72 89	138 188 413	22·7376340	8·025957
518	26 83 24	138 991 832	22·7596134	8·031129
519	26 93 61	139 798 359	22·7815715	8·036293
520	27 04 00	140 608 000	22·8035085	8·041451
521	27 14 41	141 420 761	22·8254244	8·046603
522	27 24 84	142 236 648	22·8473193	8·051748
523	27 35 29	143 055 667	22·8691933	8·056886
524	27 45 76	143 877 824	22·8910463	8·062018
525	27 56 25	144 703 125	22·9128785	8·067143
526	27 66 76	145 531 576	22·9346899	8·072262
527	27 77 29	146 363 183	22·9564806	8·077374
528	27 87 84	147 197 952	22·9782506	8·082480
529	27 98 41	148 035 889	23·0000000	8·087579
530	28 09 00	148 877 000	23·0217289	8·092672
531	28 19 61	149 721 291	23·0434372	8·097759
532	28 30 24	150 568 768	23·0651252	8·102839
533	28 40 89	151 419 437	23·0867928	8·107913
534	28 51 56	152 273 304	23·1084400	8·112980
535	28 62 25	153 130 375	23·1300670	8·118041
536	28 72 96	153 990 656	23·1516738	8·123096
537	28 83 69	154 854 153	23·1732605	8·128145
538	28 94 44	155 720 872	23·1948270	8·133187
539	29 05 21	156 590 819	23·2163735	8·138223
540	29 16 00	157 464 000	23·2379001	8·143253

Num.	Square.	Cube.	Squ. Root.	Cube Root.
541	29 26 81	158 340 421	23·2594067	8·148276
542	29 37 64	159 220 088	23·2808935	8·153294
543	29 48 49	160 103 007	23·3023604	8·158305
544	29 59 36	160 989 184	23·3238076	8·163310
545	29 70 25	161 878 625	23·3452351	8·168309
546	29 81 16	162 771 336	23·3666429	8·173302
547	29 92 09	163 667 323	23·3880311	8·178289
548	30 03 04	164 566 592	23·4093998	8·183269
549	30 14 01	165 469 149	23·4307490	8·188244
550	30 25 00	166 375 000	23·4520788	8·193213
551	30 36 01	167 284 151	23·4733892	8·198175
552	30 47 04	168 196 608	23·4946802	8·203132
553	30 58 09	169 112 377	23·5159520	8·208082
554	30 69 16	170 031 464	23·5372046	8·213027
555	30 80 25	170 953 875	23·5584380	8·217966
556	30 91 36	171 879 616	23·5796522	8·222898
557	31 02 49	172 808 693	23·6008474	8·227825
558	31 13 64	173 741 112	23·6220236	8·232746
559	31 24 81	174 676 879	23·6431808	8·237661
560	31 36 00	175 616 000	23·6643191	8·242571
561	31 47 21	176 558 481	23·6854386	8·247474
562	31 58 44	177 504 328	23·7065392	8·252371
563	31 69 69	178 453 547	23·7276210	8·257263
564	31 80 96	179 406 144	23·7486842	8·262149
565	31 92 25	180 362 125	23·7697286	8·267029
566	32 03 56	181 321 496	23·7907545	8·271904
567	32 14 89	182 284 263	23·8117618	8·276773
568	32 26 24	183 250 432	23·8327506	8·281635
569	32 37 61	184 220 009	23·8537209	8·286493
570	32 49 00	185 193 000	23·8746728	8·291344
571	32 60 41	186 169 411	23·8956063	8·296190
572	32 71 84	187 149 248	23·9165215	8·301030
573	32 83 29	188 132 517	23·9374184	8·305865
574	32 94 76	189 119 224	23·9582971	8·310694
575	33 06 25	190 109 375	23·9791576	8·315517
576	33 17 76	191 102 976	24·0000000	8·320335
577	33 29 29	192 100 033	24·0208243	8·325147
578	33 40 84	193 100 552	24·0416306	8·329954
579	33 52 41	194 104 539	24·0624188	8·334755
580	33 64 00	195 112 000	24·0831891	8·339551
581	33 75 61	196 122 941	24·1039416	8·344341
582	33 87 24	197 137 368	24·1246762	8·349126
583	33 98 89	198 155 287	24·1453929	8·353905
584	34 10 56	199 176 704	24·1660919	8·358678
585	34 22 25	200 201 625	24·1867732	8·363447

Num.	Square.	Cube.	Squ. Root.	Cube Root.
586	34 33 96	201 230 056	24·2074369	8·368209
587	34 45 69	202 262 003	24·2280829	8·372967
588	34 57 44	203 297 472	24·2487113	8·377719
589	34 69 21	204 336 469	24·2693222	8·382465
590	34 81 00	205 379 000	24·2899156	8·387206
591	34 92 81	206 425 071	24·3104916	8·391942
592	35 04 64	207 474 688	24·3310501	8·396673
593	35 16 49	208 527 857	24·3515913	8·401398
594	35 28 36	209 584 584	24·3721152	8·406118
595	35 40 25	210 644 875	24·3926218	8·410833
596	35 52 16	211 708 736	24·4131112	8·415542
597	35 64 09	212 776 173	24·4335834	8·420246
598	35 76 04	213 847 192	24·4540385	8·424945
599	35 88 01	214 921 799	24·4744765	8·429638
600	36 00 00	216 000 000	24·4948974	8·434327
601	36 12 01	217 081 801	24·5153013	8·439010
602	36 24 04	218 167 208	24·5356883	8·443688
603	36 36 09	219 256 227	24·5560583	8·448360
604	36 48 16	220 348 864	24·5764115	8·453028
605	36 60 25	221 445 125	24·5967478	8·457691
606	36 72 36	222 545 016	24·6170673	8·462348
607	36 84 49	223 648 543	24·6373700	8·467000
608	36 96 64	224 755 712	24·6576560	8·471647
609	37 08 81	225 866 529	24·6779254	8·476289
610	37 21 00	226 981 000	24·6981781	8·480926
611	37 33 21	228 099 131	24·7184142	8·485558
612	37 45 44	229 220 928	24·7386338	8·490185
613	37 57 69	230 346 397	24·7588368	8·494806
614	37 69 96	231 475 544	24·7790234	8·499423
615	37 82 25	232 608 375	24·7991935	8·504035
616	37 94 56	233 744 896	24·8193473	8·508642
617	38 06 89	234 885 113	24·8394847	8·513243
618	38 19 24	236 029 032	24·8596058	8·517840
619	38 31 61	237 176 659	24·8797106	8·522432
620	38 44 00	238 328 000	24·8997992	8·527019
621	38 56 41	239 483 061	24·9198716	8·531601
622	38 68 84	240 641 848	24·9399278	8·536178
623	38 81 29	241 804 367	24·9599679	8·540750
624	38 93 76	242 970 624	24·9799920	8·545317
625	39 06 25	244 140 625	25·0000000	8·549880
626	39 18 76	245 314 376	25·0199920	8·554437
627	39 31 29	246 491 883	25·0399681	8·558990
628	39 43 84	247 673 152	25·0599282	8·563538
629	39 56 41	248 858 189	25·0798724	8·568081
630	39 69 00	250 047 000	25·0998008	8·572619

Num.	Square.	Cube.	Squ. Root.	Cube Root.
631	39 81 61	251 239 591	25·1197134	8·577152
632	39 94 24	252 435 968	25·1396102	8·581681
633	40 06 89	253 636 137	25·1594913	8·586205
634	40 19 56	254 840 104	25.1793566	8·590724
635	40 32 25	256 047 875	25·1992063	8·595238
636	40 44 96	257 259 456	25·2190404	8·599748
637	40 57 69	258 474 853	25·2388589	8·604252
638	40 70 44	259 694 072	25·2586619	8·608753
639	40 83 21	260 917 119	25·2784493	8·613248
640	40 96 00	262 144 000	25·2982213	8·617739
641	41 08 81	263 374 721	25·3179778	8·622225
642	41 21 64	264 609 288	25·3377189	8·626706
643	41 34 49	265 847 707	25·3574447	8·631183
644	41 47 36	267 089 984	25·3771551	8·635655
645	41 60 25	268 336 125	25·3968502	8·640123
646	41 73 16	269 586 136	25·4165301	8·644585
647	41 86 09	270 840 023	25·4361947	8·649044
648	41 99 04	272 097 792	25·4558441	8·653497
649	42 12 01	273 359 449	25·4754784	8·657946
650	42 25 00	274 625 000	25·4950976	8·662391
651	42 38 01	275 894 451	25·5147016	8·666831
652	42 51 04	277 167 808	25·5342907	8·671266
653	42 64 09	278 445 077	25·5538647	8·675697
654	42 77 16	279 726 264	25·5734237	8·680124
655	42 90 25	281 011 375	25·5929678	8·684546
656	43 03 36	282 300 416	25·6124969	8·688963
657	43 16 49	283 593 393	25·6320112	8·693376
658	43 29 64	284 890 312	25·6515107	8·697784
659	43 42 81	286 191 179	25·6709953	8·702188
660	43 56 00	287 496 000	25·6904652	8·706588
661	43 69 21	288 804 781	25·7099203	8·710983
662	43 82 44	290 117 528	25·7293607	8·715373
663	43 95 69	291 434 247	25·7487864	8·719760
664	44 08 96	292 754 944	25·7681975	8·724141
665	44 22 25	294 079 625	25·7875939	8·728519
666	44 35 56	295 408 296	25·8069758	8·732892
667	44 48 89	296 740 963	25·8263431	8·737260
668	44 62 24	298 077 632	25·8456960	8·741625
669	44 75 61	299 418 309	25·8650343	8·745985
670	44 89 00	300 763 000	25·8843582	8·750340
671	45 02 41	302 111 711	25·9036677	8·754691
672	45 15 84	303 464 448	25·9229628	8·759038
673	45 29 29	304 821 217	25·9422435	8·763381
674	45 42 76	306 182 024	25·9615100	8·767719
675	45 56 25	307 546 875	25·9807621	8·772053

Num.	Square.	Cube.	Squ. Root.	Cube Root.
676	45 69 76	308 915 776	26·0000000	8·776383
677	45 83 29	310 288 733	26·0192237	8·780708
678	45 96 84	311 665 752	26·0384331	8·785030
679	46 10 41	313 046 839	26·0576284	8·789347
680	46 24 00	314 432 000	26·0768096	8·793659
681	46 37 61	315 821 241	26·0959767	8·797968
682	46 51 24	317 214 568	26·1151297	8·802272
683	46 64 89	318 611 987	26·1342687	8·806572
684	46 78 56	320 013 504	26·1533937	8·810868
685	46 92 25	321 419 125	26·1725047	8·815160
686	47 05 96	322 828 856	26·1916017	8·819447
687	47 19 69	324 242 703	26·2106848	8·823731
688	47 33 44	325 660 672	26·2297541	8·828010
689	47 47 21	327 082 769	26·2488095	8·832285
690	47 61 00	328 509 000	26·2678511	8·836556
691	47 74 81	329 939 371	26·2868789	8·840823
692	47 88 64	331 373 888	26·3058929	8·845085
693	48 02 49	332 812 557	26·3248932	8·849344
694	48 16 36	334 255 384	26·3438797	8·853598
695	48 30 25	335 702 375	26·3628527	8·857849
696	48 44 16	337 153 536	26·3818119	8·862095
697	48 58 09	338 608 873	26·4007576	8·866337
698	48 72 04	340 068 392	26·4196896	8·870576
699	48 86 01	341 532 099	26·4386081	8·874810
700	49 00 00	343 000 000	26·4575131	8·879040
701	49 14 01	344 472 101	26·4764046	8·883266
702	49 28 04	345 948 408	26·4952826	8·887488
703	49 42 09	347 428 927	26·5141472	8·891706
704	49 56 16	348 913 664	26·5329983	8·895920
705	49 70 25	350 402 625	26·5518361	8·900130
706	49 84 36	351 895 816	26·5706605	8·904337
707	49 98 49	353 393 243	26·5894716	8·908539
708	50 12 64	354 894 912	26·6082694	8·912737
709	50 26 81	356 400 829	26·6270539	8·916931
710	50 41 00	357 911 000	26·6458252	8·921121
711	50 55 21	359 425 431	26·6645833	8·925308
712	50 69 44	360 944 128	26·6833281	8·929490
713	50 83 69	362 467 097	26·7020598	8·933669
714	50 97 96	363 994 344	26·7207784	8·937843
715	51 12 25	365 525 875	26·7394839	8·942014
716	51 26 56	367 061 696	26·7581763	8·946181
717	51 40 89	368 601 813	26·7768557	8·950344
718	51 55 24	370 146 232	26·7955220	8·954503
719	51 69 61	371 694 959	26·8141754	8·958658
720	51 84 00	373 248 000	26·8328157	8·962809

Num.	Square.	Cube.	Squ. Root.	Cube Root.
721	51 98 41	374 805 361	26·8514432	8·966957
722	52 12 84	376 367 048	26·8700577	8·971101
723	52 27 29	377 933 067	26·8886593	8·975241
724	52 41 76	379 503 424	26·9072481	8·979377
725	52 56 25	381 078 125	26·9258240	8·983509
726	52 70 76	382 657 176	26·9443872	8·987637
727	52 85 29	384 240 583	26·9629375	8·991762
728	52 99 84	385 828 352	26·9814751	8·995883
729	53 14 41	387 420 489	27·0000000	9·000000
730	53 29 00	389 017 000	27·0185122	9·004113
731	53 43 61	390 617 891	27·0370117	9·008223
732	53 58 24	392 223 168	27·0554985	9·012329
733	53 72 89	393 832 837	27·0739727	9·016431
734	53 87 56	395 446 904	27·0924344	9·020529
735	54 02 25	397 065 375	27·1108834	9·024624
736	54 16 96	398 688 256	27·1293199	9·028715
737	54 31 69	400 315 553	27·1477439	9·032802
738	54 46 44	401 947 272	27·1661554	9·036886
739	54 61 21	403 583 419	27·1845544	9·040965
740	54 76 00	405 224 000	27·2029410	9·045042
741	54 90 81	406 869 021	27·2213152	9·049114
742	55 05 64	408 518 488	27·2396769	9·053183
743	55 20 49	410 172 407	27·2580263	9·057248
744	55 35 36	411 830 784	27·2763634	9·061310
745	55 50 25	413 493 625	27·2946881	9·065368
746	55 65 16	415 160 936	27·3130006	9·069422
747	55 80 09	416 832 723	27·3313007	9·073473
748	55 95 04	418 508 992	27·3495887	9·077520
749	56 10 01	420 189 749	27·3678644	9·081563
750	56 25 00	421 875 000	27·3861279	9·085603
751	56 40 01	423 564 751	27·4043792	9·089639
752	56 55 04	425 259 008	27·4226184	9·093672
753	56 70 09	426 957 777	27·4408455	9·097701
754	56 85 16	428 661 064	27·4590604	9·101726
755	57 00 25	430 368 875	27·4772633	9·105748
756	57 15 36	432 081 216	27·4954542	9·109767
757	57 30 49	433 798 093	27·5136330	9·113782
758	57 45 64	435 519 512	27·5317998	9·117793
759	57 60 81	437 245 479	27·5499546	9·121801
760	57 76 00	438 976 000	27·5680975	9·125805
761	57 91 21	440 711 081	27·5862284	9·129806
762	58 06 44	442 450 728	27·6043475	9·133803
763	58 21 69	444 194 947	27·6224546	9·137797
764	58 36 96	445 943 744	27·6405499	9·141787
765	58 52 25	447 697 125	27·6586334	9·145774

Num.	Square.	Cube.	Squ. Root.	Cube Root.
766	58 67 56	449 455 096	27·6767050	9·149758
767	58 82 89	451 217 663	27·6947648	9·153737
768	58 98 24	452 984 832	27·7128129	9·157714
769	59 13 61	454 756 609	27·7308492	9·161687
770	59 29 00	456 533 000	27·7488739	9·165656
771	59 44 41	458 314 011	27·7668868	9·169622
772	59 59 84	460 099 648	27·7848880	9·173585
773	59 75 29	461 889 917	27·8028775	9·177544
774	59 90 76	463 684 824	27·8208555	9·181500
775	60 06 25	465 484 375	27·8388218	9·185453
776	60 21 76	467 288 576	27·8567766	9·189402
777	60 37 29	469 097 433	27·8747197	9·193347
778	60 52 84	470 910 952	27·8926514	9·197290
779	60 68 41	472 729 139	27·9105715	9·201229
780	60 84 00	474 552 000	27·9284801	9·205164
781	60 99 61	476 379 541	27·9463772	9·209096
782	61 15 24	478 211 768	27·9642629	9·213025
783	61 30 89	480 048 687	27·9821372	9·216950
784	61 46 56	481 890 304	28·0000000	9·220873
785	61 62 25	483 736 625	28·0178515	9·224791
786	61 77 96	485 587 656	28·0356915	9·228707
787	61 93 69	487 443 403	28·0535203	9·232619
788	62 09 44	489 303 872	28·0713377	9·236528
789	62 25 21	491 169 069	28·0891438	9·240433
790	62 41 00	493 039 000	28·1069386	9·244335
791	62 56 81	494 913 671	28·1247222	9·248234
792	62 72 64	496 793 088	28·1424946	9·252130
793	62 88 49	498 677 257	28·1602557	9·256022
794	63 04 36	500 566 184	28·1780056	9·259911
795	63 20 25	502 459 875	28·1957444	9·263797
796	63 36 16	504 358 336	28·2134720	9·267680
797	63 52 09	506 261 573	28·2311884	9·271559
798	63 68 04	508 169 592	28·2488938	9·275435
799	63 84 01	510 082 399	28·2665881	9·279308
800	64 00 00	512 000 000	28·2842712	9·283178
801	64 16 01	513 922 401	28·3019434	9·287044
802	64 32 04	515 849 608	28·3196045	9·290907
803	64 48 09	517 781 627	28·3372546	9·294767
804	64 64 16	519 718 464	28·3548938	9·298624
805	64 80 25	521 660 125	28·3725219	9·302477
806	64 96 36	523 606 616	28·3901391	9·306328
807	65 12 49	525 557 943	28·4077454	9·310175
808	65 28 64	527 514 112	28·4253408	9·314019
809	65 44 81	529 475 129	28·4429253	9·317860
810	65 61 00	531 441 000	28·4604989	9·321697

Num.	Square.	Cube.	Squ. Root.	Cube Root.
811	657721	533411731	28·4780617	9·325532
812	659344	535387328	28·4956137	9·329363
813	660969	537367797	28·5131549	9·333192
814	662596	539353144	28·5306852	9·337017
815	664225	541343375	28·5482048	9·340839
816	665856	543338496	28·5657137	9·344657
817	667489	545338513	28·5832119	9·348473
818	669124	547343432	28·6006993	9·352286
819	670761	549353259	28·6181760	9·356095
820	672400	551368000	28·6356421	9·359902
821	674041	553387661	28·6530976	9·363705
822	675684	555412248	28·6705424	9·367505
823	677329	557441767	28·6879766	9·371302
824	678976	559476224	28·7054002	9·375096
825	680625	561515625	28·7228132	9·378887
826	682276	563559976	28·7402157	9·382675
827	683929	565609283	28·7576077	9·386460
828	685584	567663552	28·7749891	9·390242
829	687241	569722789	28·7923601	9·394021
830	688900	571787000	28·8097206	9·397796
831	690561	573856191	28·8270706	9·401569
832	692224	575930368	28·8444102	9·405339
833	693889	578009537	28·8617394	9·409105
834	695556	580093704	28·8790582	9·412869
835	697225	582182875	28·8963666	9·416630
836	698896	584277056	28·9136646	9·420387
837	700569	586376253	28·9309523	9·424142
838	702244	588480472	28·9482297	9·427894
839	703921	590589719	28·9654967	9·431642
840	705600	592704000	28·9827535	9·435388
841	707281	594823321	29·0000000	9·439131
842	708964	596947688	29·0172363	9·442870
843	710649	599077107	29·0344623	9·446607
844	712336	601211584	29·0516781	9·450341
845	714025	603351125	29·0688837	9·454072
846	715716	605495736	29·0860791	9·457800
847	717409	607645423	29·1032644	9·461525
848	719104	609800192	29·1204396	9·465247
849	720801	611960049	29·1376046	9·468966
850	722500	614125000	29·1547595	9·472682
851	724201	616295051	29·1719043	9·476396
852	725904	618470208	29·1890390	9·480106
853	727609	620650477	29·2061637	9·483814
854	729316	622835864	29·2232784	9·487518
855	731025	625026375	29·2403830	9·491220

Num.	Square.	Cube.	Squ. Root.	Cube Root.
856	73 27 36	627 222 016	29·2574777	9·494919
857	73 44 49	629 422 793	29·2745623	9·498615
858	73 61 64	631 628 712	29·2916370	9·502308
859	73 78 81	633 839 779	29·3087018	9·505998
860	73 96 00	636 056 000	29·3257566	9·509685
861	74 13 21	638 277 381	29·3428015	9·513370
862	74 30 44	640 503 928	29·3598365	9·517051
863	74 47 69	642 735 647	29·3768616	9·520730
864	74 64 96	644 972 544	29·3938769	9·524406
865	74 82 25	647 214 625	29·4108823	9·528079
866	74 99 56	649 461 896	29·4278779	9·531750
867	75 16 89	651 714 363	29·4448637	9·535417
868	75 34 24	653 972 032	29·4618397	9·539082
869	75 51 61	656 234 909	29·4788059	9·542744
870	75 69 00	658 503 000	29·4957624	9·546403
871	75 86 41	660 776 311	29·5127091	9·550059
872	76 03 84	663 054 848	29·5296461	9·553712
873	76 21 29	665 338 617	29·5465734	9·557363
874	76 38 76	667 627 624	29·5634910	9·561011
875	76 56 25	669 921 875	29·5803989	9·564656
876	76 73 76	672 221 376	29·5972972	9·568298
877	76 91 29	674 526 133	29·6141858	9·571938
878	77 08 84	676 836 152	29·6310648	9·575574
879	77 26 41	679 151 439	29·6479342	9·579208
880	77 44 00	681 472 000	29·6647939	9·582840
881	77 61 61	683 797 841	29·6816442	9·586468
882	77 79 24	686 128 968	29·6984848	9·590094
883	77 96 89	688 465 387	29·7153159	9·593717
884	78 14 56	690 807 104	29·7321375	9·597337
885	78 32 25	693 154 125	29·7489496	9·600955
886	78 49 96	695 506 456	29·7657521	9·604570
887	78 67 69	697 864 103	29·7825452	9·608182
888	78 85 44	700 227 072	29·7993289	9·611791
889	79 03 21	702 595 369	29·8161030	9·615398
890	79 21 00	704 969 000	29·8328678	9·619002
891	79 38 81	707 347 971	29·8496231	9·622603
892	79 56 64	709 732 288	29·8663690	9·626202
893	79 74 49	712 121 957	29·8831056	9·629797
894	79 92 36	714 516 984	29·8998328	9·633391
895	80 10 25	716 917 375	29·9165506	9·636981
896	80 28 16	719 323 136	29·9332591	9·640569
897	80 46 09	721 734 273	29·9499583	9·644154
898	80 64 04	724 150 792	29·9666481	9·647737
899	80 82 01	726 572 699	29·9833287	9·651317
900	81 00 00	729 000 000	30·0000000	9·654894

Num.	Square.	Cube.	Squ. Root.	Cube Root.
901	81 18 01	731 432 701	30·0166620	9·658468
902	81 36 04	733 870 808	30·0333148	9·662040
903	81 54 09	736 314 327	30·0499584	9·665610
904	81 72 16	738 763 264	30·0665928	9·669176
905	81 90 25	741 217 625	30·0832179	9·672740
906	82 08 36	743 677 416	30·0998339	9·676302
907	82 26 49	746 142 643	30·1164407	9·679860
908	82 44 64	748 613 312	30·1330383	9·683417
909	82 62 81	751 089 429	30·1496269	9·686970
910	82 81 00	753 571 000	30·1662063	9·690521
911	82 99 21	756 058 031	30·1827765	9·694069
912	83 17 44	758 550 528	30·1993377	9·697615
913	83 35 69	761 048 497	30·2158899	9·701158
914	83 53 96	763 551 944	30·2324329	9·704699
915	83 72 25	766 060 875	30·2489669	9·708237
916	83 90 56	768 575 296	30·2654919	9·711772
917	84 08 89	771 095 213	30·2820079	9·715305
918	84 27 24	773 620 632	30·2985148	9·718835
919	84 45 61	776 151 559	30·3150128	9·722363
920	84 64 00	778 688 000	30·3315018	9·725888
921	84 82 41	781 229 961	30·3479818	9·729411
922	85 00 84	783 777 448	30·3644529	9·732931
923	85 19 29	786 330 467	30·3809151	9·736448
924	85 37 76	788 889 024	30·3973683	9·739963
925	85 56 25	791 453 125	30·4138127	9·743476
926	85 74 76	794 022 776	30·4302481	9·746986
927	85 93 29	796 597 983	30·4466747	9·750493
928	86 11 84	799 178 752	30·4630924	9·753998
929	86 30 41	801 765 089	30·4795013	9·757500
930	86 49 00	804 357 000	30·4959014	9·761000
931	86 67 61	806 954 491	30·5122926	9·764497
932	86 86 24	809 557 568	30·5286750	9·767992
933	87 04 89	812 166 237	30·5450487	9·771484
934	87 23 56	814 780 504	30·5614136	9·774974
935	87 42 25	817 400 375	30·5777697	9·778462
936	87 60 96	820 025 856	30·5941171	9·781947
937	87 79 69	822 656 953	30·6104557	9·785429
938	87 98 44	825 293 672	30·6267857	9·788909
939	88 17 21	827 936 019	30·6431069	9·792386
940	88 36 00	830 584 000	30·6594194	9·795861
941	88 54 81	833 237 621	30·6757233	9·799334
942	88 73 64	835 896 888	30·6920185	9·802804
943	88 92 49	838 561 807	30·7083051	9·806271
944	89 11 36	841 232 384	30·7245830	9·809736
945	89 30 25	843 908 625	30·7408523	9·813199

Num.	Square.	Cube.	Squ. Root.	Cube Root.
946	89 49 16	846 590 536	30·7571130	9·816659
947	89 68 09	849 278 123	30·7733651	9·820117
948	89 87 04	851 971 392	30·7896086	9·823572
949	90 06 01	854 670 349	30·8058436	9·827025
950	90 25 00	857 375 000	30·8220700	9·830476
951	90 44 01	860 085 351	30·8382879	9·833924
952	90 63 04	862 801 408	30·8544972	9·837369
953	90 82 09	865 523 177	30·8706981	9·840813
954	91 01 16	868 250 664	30·8868904	9·844254
955	91 20 25	870 983 875	30·9030743	9·847692
956	91 39 36	873 722 816	30·9192497	9·851128
957	91 58 49	876 467 493	30·9354166	9·854562
958	91 77 64	879 217 912	30·9515751	9·857993
959	91 96 81	881 974 079	30·9677251	9·861422
960	92 16 00	884 736 000	30·9838668	9·864848
961	92 35 21	887 503 681	31·0000000	9·868272
962	92 54 44	890 277 128	31·0161248	9·871694
963	92 73 69	893 056 347	31·0322413	9·875113
964	92 92 96	895 841 344	31·0483494	9·878530
965	93 12 25	898 632 125	31·0644491	9·881945
966	93 31 56	901 428 696	31·0805405	9·885357
967	93 50 89	904 231 063	31·0966236	9·888767
968	93 70 24	907 039 232	31·1126984	9·892175
969	93 89 61	909 853 209	31·1287648	9·895580
970	94 09 00	912 673 000	31·1448230	9·898983
971	94 28 41	915 498 611	31·1608729	9·902384
972	94 47 84	918 330 048	31·1769145	9·905782
973	94 67 29	921 167 317	31·1929479	9·909178
974	94 86 76	924 010 424	31·2089731	9·912571
975	95 06 25	926 859 375	31·2249900	9·915962
976	95 25 76	929 714 176	31·2409987	9·919351
977	95 45 29	932 574 833	31·2569992	9·922738
978	95 64 84	935 441 352	31·2729915	9·926122
979	95 84 41	938 313 739	31·2889757	9·929504
980	96 04 00	941 192 000	31·3049517	9·932884
981	96 23 61	944 076 141	31·3209195	9·936261
982	96 43 24	946 966 168	31·3368792	9·939636
983	96 62 89	949 862 087	31·3528308	9·943009
984	96 82 56	952 763 904	31·3687743	9·946380
985	97 02 25	955 671 625	31·3847097	9·949748
986	97 21 96	958 585 256	31·4006369	9·953114
987	97 41 69	961 504 803	31·4165561	9·956477
988	97 61 44	964 430 272	31·4324673	9·959839
989	97 81 21	967 361 669	31·4483704	9·963198
990	98 01 00	970 299 000	31·4642654	9·966555

Num.	Square.	Cube.	Squ. Root.	Cube Root.
991	98 20 81	973 242 271	31·4801525	9·969909
992	98 40 64	976 191 488	31·4960315	9·973262
993	98 60 49	979 146 657	31·5119025	9·976612
994	98 80 36	982 107 784	31·5277655	9·979960
995	99 00 25	985 074 875	31·5436206	9·983305
996	99 20 16	988 047 936	31·5594677	9·986649
997	99 40 09	991 026 973	31·5753068	9·989990
998	99 60 04	994 011 992	31·5911380	9·993329
999	99 80 01	997 002 999	31·6069613	9·996666
1000	1 00 00 00	1 000 000 000	31·6227766	10·000000

TABLE

OF

CIRCUMFERENCES AND AREAS OF CIRCLES,

CORRESPONDING TO

DIAMETERS FOR EVERY QUARTER OF THE UNIT

BETWEEN 1 AND 100.

Diam.	Circumf.	Area.	Diam.	Circumf.	Area.
1.00	3.1416	0.7854	12.00	37.6991	113.0973
1.25	3.9270	1.2272	12.25	38.4845	117.8588
1.50	4.7124	1.7671	12.50	39.2699	122.7185
1.75	5.4978	2.4053	12.75	40.0553	127.6763
2.00	6.2832	3.1416	13.00	40.8407	132.7323
2.25	7.0686	3.9761	13.25	41.6261	137.8865
2.50	7.8540	4.9087	13.50	42.4115	143.1388
2.75	8.6394	5.9396	13.75	43.1969	148.4893
3.00	9.4248	7.0686	14.00	43.9823	153.9380
3.25	10.2102	8.2958	14.25	44.7677	159.4849
3.50	10.9956	9.6211	14.50	45.5531	165.1300
3.75	11.7810	11.0447	14.75	46.3385	170.8732
4.00	12.5664	12.5664	15.00	47.1239	176.7146
4.25	13.3518	14.1863	15.25	47.9093	182.6542
4.50	14.1372	15.9043	15.50	48.6947	188.6919
4.75	14.9226	17.7205	15.75	49.4801	194.8278
5.00	15.7080	19.6350	16.00	50.2655	201.0619
5.25	16.4934	21.6475	16.25	51.0509	207.3942
5.50	17.2788	23.7583	16.50	51.8363	213.8246
5.75	18.0642	25.9672	16.75	52.6217	220.3533
6.00	18.8496	28.2743	17.00	53.4071	226.9801
6.25	19.6350	30.6796	17.25	54.1925	233.7050
6.50	20.4204	33.1831	17.50	54.9779	240.5282
6.75	21.2058	35.7847	17.75	55.7633	247.4495
7.00	21.9911	38.4845	18.00	56.5487	254.4690
7.25	22.7765	41.2825	18.25	57.3341	261.5867
7.50	23.5619	44.1786	18.50	58.1195	268.8025
7.75	24.3473	47.1730	18.75	58.9049	276.1165
8.00	25.1327	50.2655	19.00	59.6903	283.5287
8.25	25.9181	53.4562	19.25	60.4757	291.0391
8.50	26.7035	56.7450	19.50	61.2611	298.6477
8.75	27.4889	60.1320	19.75	62.0465	306.3544
9.00	28.2743	63.6173	20.00	62.8319	314.1593
9.25	29.0597	67.2006	20.25	63.6173	322.0623
9.50	29.8451	70.8822	20.50	64.4026	330.0636
9.75	30.6305	74.6619	20.75	65.1880	338.1630
10.00	31.4159	78.5398	21.00	65.9734	346.3606
10.25	32.2013	82.5159	21.25	66.7588	354.6564
10.50	32.9867	86.5901	21.50	67.5442	363.0503
10.75	33.7721	90.7626	21.75	68.3296	371.5424
11.00	34.5575	95.0332	22.00	69.1150	380.1327
11.25	35.3429	99.4020	22.25	69.9004	388.8212
11.50	36.1283	103.8689	22.50	70.6858	397.6078
11.75	36.9137	108.4340	22.75	71.4712	406.4926

Diam.	Circumf.	Area.	Diam.	Circumf.	Area.
23.00	72.2566	415.4756	34.00	106.8142	907.9203
23.25	73.0420	424.5568	34.25	107.5995	921.3211
23.50	73.8274	433.7361	34.50	108.3849	934.8202
23.75	74.6128	443.0137	34.75	109.1703	948.4174
24.00	75.3982	452.3893	35.00	109.9557	962.1128
24.25	76.1836	461.8632	35.25	110.7411	975.9063
24.50	76.9690	471.4352	35.50	111.5265	989.7980
24.75	77.7544	481.1055	35.75	112.3119	1003.7879
25.00	78.5398	490.8739	36.00	113.0973	1017.8760
25.25	79.3252	500.7404	36.25	113.8827	1032.0623
25.50	80.1106	510.7052	36.50	114.6681	1046.3467
25.75	80.8960	520.7681	36.75	115.4535	1060.7293
26.00	81.6814	530.9292	37.00	116.2389	1075.2101
26.25	82.4668	541.1884	37.25	117.0243	1089.7890
26.50	83.2522	551.5459	37.50	117.8097	1104.4662
26.75	84.0376	562.0015	37.75	118.5957	1119.2415
27.00	84.8230	572.5553	38.00	119.3805	1134.1149
27.25	85.6084	583.2072	38.25	120.1659	1149.0866
27.50	86.3938	593.9574	38.50	120.9513	1164.1564
27.75	87.1792	604.8057	38.75	121.7367	1179.3244
28.00	87.9646	615.7522	39.00	122.5221	1194.5906
28.25	88.7500	626.7968	39.25	123.3075	1209.9550
28.50	89.5354	637.9397	39.50	124.0929	1225.4175
28.75	90.3208	649.1807	39.75	124.8783	1240.9782
29.00	91.1062	660.5199	40.00	125.6637	1256.6371
29.25	91.8916	671.9572	40.25	126.4491	1272.3941
29.50	92.6770	683.4928	40.50	127.2345	1288.2493
29.75	93.4624	695.1265	40.75	128.0199	1304.2027
30.00	94.2478	706.8583	41.00	128.8053	1320.2543
30.25	95.0332	718.6884	41.25	129.5907	1336.4041
30.50	95.8186	730.6166	41.50	130.3761	1352.6520
30.75	96.6040	742.6431	41.75	131.1615	1368.9981
31.00	97.3894	754.7676	42.00	131.9469	1385.4424
31.25	98.1748	766.9904	42.25	132.7323	1401.9848
31.50	98.9602	779.3113	42.50	133.5177	1418.6254
31.75	99.7456	791.7304	42.75	134.3031	1435.3642
32.00	100.5310	804.2477	43.00	135.0835	1452.2012
32.25	101.3164	816.8632	43.25	135.8739	1469.1364
32.50	102.1018	829.5768	43.50	136.6593	1486.1697
32.75	102.8872	842.3886	43.75	137.4447	1503.3012
33.00	103.6726	855.2986	44.00	138.2301	1520.5308
33.25	104.4580	868.3068	44.25	139.0155	1537.8587
33.50	105.2434	881.4131	44.50	139.8009	1556.2847
33.75	106.0288	894.6176	44.75	140.5863	1574.8089

Diam.	Circumf.	Area.	Diam.	Circumf.	Area.
45·00	141·3717	1590·4313	56·00	175·9292	2463·0086
45·25	142·1571	1608·1518	56·25	176·7146	2485·0489
45·50	142·9425	1625·9705	56·50	177·5000	2507·1873
45·75	143·7279	1643·8874	56·75	178·2854	2529·4239
46·00	144·5133	1661·9025	57·00	179·0708	2551·7586
46·25	145·2987	1680·0158	57·25	179·8562	2574·1916
46·50	146·0841	1698·2272	57·50	180·6416	2596·7227
46·75	146·8695	1716·5368	57·75	181·4270	2619·3520
47·00	147·6549	1734·9445	58·00	182·2124	2642·0794
47·25	148·4403	1753·4505	58·25	182·9978	2664·9051
47·50	149·2257	1772·0546	58·50	183·7832	2687·8289
47·75	150·0110	1790·7569	58·75	184·5686	2710·8508
48·00	150·7964	1809·5574	59·00	185·3540	2733·9710
48·25	151·5818	1828·4560	59·25	186·1394	2757·1893
48·50	152·3672	1847·4528	59·50	186·9248	2780·5058
48·75	153·1526	1866·5478	59·75	187·7102	2803·9206
49·00	153·9380	1885·7410	60·00	188·4956	2827·4334
49·25	154·7234	1905·0323	60·25	189·2810	2851·0444
49·50	155·5088	1924·4218	60·50	190·0664	2874·7536
49·75	156·2942	1943·9095	60·75	190·8518	2898·5610
50·00	157·0796	1963·4954	61·00	191·6372	2922·4666
50·25	157·8650	1983·1794	61·25	192·4226	2946·4703
50·50	158·6504	2002·9617	61·50	193·2079	2970·5722
50·75	159·4358	2022·8421	61·75	193·9933	2994·7723
51·00	160·2212	2042·8206	62·00	194·7787	3019·0705
51·25	161·0066	2062·8974	62·25	195·5641	3043·4670
51·50	161·7920	2083·0723	62·50	196·3495	3067·9616
51·75	162·5774	2103·3454	62·75	197·1349	3092·5544
52·00	163·3628	2123·7166	63·00	197·9203	3117·2453
52·25	164·1482	2144·1861	63·25	198·7057	3142·0344
52·50	164·9336	2164·7537	63·50	199·4911	3166·9217
52·75	165·7190	2185·4195	63·75	200·2765	3191·9072
53·00	166·5044	2206·1834	64·00	201·0619	3216·9909
53·25	167·2898	2227·0456	64·25	201·8473	3242·1727
53·50	168·0752	2248·0059	64·50	202·6327	3267·4527
53·75	168·8606	2269·0644	64·75	203·4181	3292·8309
54·00	169·6460	2290·2210	65·00	204·2035	3318·3072
54·25	170·4314	2311·4759	65·25	204·9889	3343·8818
54·50	171·2168	2332·8289	65·50	205·7743	3369·5545
54·75	172·0022	2354·2801	65·75	206·5597	3395·3253
55·00	172·7876	2375·8294	66·00	207·3451	3421·1944
55·25	173·5730	2397·4770	66·25	208·1305	3447·1616
55·50	174·3584	2419·2227	66·50	208·9159	3473·2270
55·75	175·1438	2441·0666	66·75	209·7013	3499·3906

Diam.	Circumf.	Area.	Diam.	Circumf.	Area.
67·00	210·4867	3525·6524	78·00	245·0442	4778·3624
67·25	211·2721	3552·0123	78·25	245·8296	4809·0420
67·50	212·0575	3578·4704	78·50	246·6150	4839·8198
67·75	212·8429	3605·0267	78·75	247·4004	4870·7958
68·00	213·6283	3631·6811	79·00	248·1858	4901·6699
68·25	214·4137	3658·4337	79·25	248·9712	4932·7422
68·50	215·1991	3685·2845	79·50	249·7566	4963·9127
68·75	215·9845	3712·2335	79·75	250·5420	4995·1814
69·00	216·7699	3739·2807	80·00	251·3274	5026·5482
69·25	217·5553	3766·4260	80·25	252·1128	5058·0132
69·50	218·3407	3793·6695	80·50	252·8982	5089·5764
69·75	219·1261	3821·0112	80·75	253·6836	5121·2378
70·00	219·9115	3848·4510	81·00	254·4690	5152·9974
70·25	220·6969	3875·9890	81·25	255·2544	5184·8551
70·50	221·4823	3903·6252	81·50	256·0398	5216·8110
70·75	222·2677	3931.3596	81·75	256·8252	5248·8650
71·00	223·0531	3959·1921	82·00	257·6106	5281·0173
71·25	223·8385	3987·1229	82·25	258·3960	5313·2677
71·50	224·6239	4015·1518	82·50	259·1814	5345·6162
71·75	225·4093	4043·2788	82·75	259·9668	5378·0630
72·00	226·1947	4071·5041	83·00	260·7522	5410·6079
72·25	226·9801	4099·8275	83·25	261·5376	5443·2510
72·50	227·7655	4128·2491	83·50	262·3230	5475·9923
72·75	228·5509	4156·7689	83·75	263·1084	5508·8318
73·00	229·3363	4185·3868	84·00	263·8938	5541·7694
73·25	230·1217	4214·1029	84·25	264·6792	5574·8052
73·50	230·9071	4242·9172	84·50	265·4646	5607·9392
73·75	231·6925	4271·8297	84·75	266·2500	5641·1714
74·00	232·4779	4300·8403	85·00	267·0354	5674·5017
74·25	233·2633	4329·9492	85·25	267·8208	5707·9302
74·50	234·0487	4359·1562	85·50	268·6062	5741·4569
74·75	234·8341	4388·4613	85·75	269·3916	5775·0818
75·00	235·6194	4417·8647	86·00	270·1770	5808·8048
75·25	236·4048	4447·3662	86·25	270·9624	5842·6260
75·50	237·1902	4476·9659	86·50	271·7478	5876·5454
75·75	237·9756	4506·6637	86·75	272·5332	5910·5630
76·00	238·7610	4536·4598	87·00	273·3186	5944·6787
76·25	239·5464	4566·3540	87·25	274·1040	5978·8926
76·50	240·3318	4596·3464	87·50	274·8894	6013·2047
76·75	241·1172	4626·4370	87·75	275·6748	6047·6149
77·00	241·9026	4656·6257	88·00	276·4602	6082·1234
77·25	242·6880	4686·9126	88·25	277·2456	6116·7300
77·50	243·4734	4717·2977	88·50	278·0309	6151·4348
77·75	244·2588	4747·7810	88·75	278·8163	6186·2377

Diam.	Circumf.	Area.	Diam.	Circumf.	Area.
89·00	279·6017	6221·1389	95·00	298·4513	7088·2184
89·25	280·3871	6256·1382	95·25	299·2367	7125·5739
89·50	281·1725	6291·2356	95·50	300·0221	7163·0276
89·75	281·9579	6326·4313	95·75	300·8075	7200·5794
90·00	282·7433	6361·7251	96·00	301·5929	7238·2295
90·25	283·5287	6397·1171	96·25	302·3783	7275·9777
90·50	284·3141	6432·6073	96·50	303·1637	7313·8240
90·75	285·0995	6468·1957	96·75	303·9491	7351·7686
91·00	285·8849	6503·8822	97·00	304·7345	7389·8113
91·25	286·6703	6539·6669	97·25	305·5199	7427·9522
91·50	287·4557	6575·5498	97·50	306·3053	7466·1913
91·75	288·2411	6611·5308	97·75	307·0907	7504·5285
92·00	289·0265	6647·6101	98·00	307·8761	7542·9640
92·25	289·8119	6683·7875	98·25	308·6615	7581·4976
92·50	290·5973	6720·0630	98·50	309·4469	7620·1293
92·75	291·3827	6756·4368	98·75	310·2323	7658·8593
93·00	292·1681	6792·9087	99·00	311·0177	7697·6874
93·25	292·9535	6829·4788	99·25	311·8031	7736·6137
93·50	293·7389	6866·1471	99·50	312·5885	7775·6382
93·75	294·5243	6902·9135	99·75	313·3739	7814·7608
94·00	295·3097	6939·7782	100·00	314·1593	7853·9816
94·25	296·0951	6976·7410			
94·50	296·8805	7013·8019			
94·75	297·6659	7050·9611			

THE END.

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